ON WALSH-FOURIER SERIES

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1. Introduction. Let the Rademacher functions be defined by

$$(1.1)\qquad \begin{array}{l} \varphi_0(x) = 1(0 \leqq x < 1/2),\ \varphi_0(x) = -1(1/2 \leqq x < 1), \\ \varphi_0(x+1) = \varphi_0(x),\ \varphi_n(x) = \varphi_0(2^n x) \qquad (n = 1, 2, \cdots). \end{array}$$

Then the Walsh functions are given by

$$(1.2)\qquad \psi_0(x) \equiv 1, \psi_n(x) = \varphi_{n_1}(x)\varphi_{n_2}(x)\cdots\varphi_{n_r}(x)$$

for $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$, where the integers n_i are uniquely determined by $n_{i+1} < n_i$. As is well known, $\{\psi_n(x)\}$ form a complete orthonormal set, and every periodic function $f(x)$ which is integrable on (0, 1) can be expanded into a Walsh-Fourier series

$$(1.3)\qquad f(x) \sim c_0 + c_1\psi_1(x) + c_2\psi_2(x) + \cdots,$$

where the coefficients are given by

$$(1.4)\qquad c_n = \int_0^1 \psi_n(x)f(x)dx \qquad (n = 0, 1, 2 \cdots).$$

Recently N. J. Fine [1] has introduced the notion of "dyadic group" and shown that the Walsh functions $\{\psi_n(x)\}$ reduce to the character group of this group. Basing on this fact, he has succeeded in developing the theory of Walsh-Fourier series analogously to that of trigonometric-Fourier series. In the present paper we shall deal with the certain theorems on WFS[2], concerning the Cesàro summability, convergency, special series and the convergence factors. The results obtained here are completely analogous to those in the case of TFS.

Our proofs mostly depend on the fundamental results obtained by N. J. Fine [1], so we shall set up his results which are needed in the sequel.

1°. *The "dyadic group"*. The dyadic group G may be defined as the denumerable direct product of the group with elements 0 and 1, in which the group operation is addition modulo 2. Thus the dyadic group G is the set of all 0, 1 sequences in which the group operation, which we shall denote by $\dotplus$, is addition modulo 2 for each element.

Let $\bar{x}$ be an element of G, $\bar{x} = \{x_1, x_2, \cdots\}$, $x_n = 0, 1$. We define the function

$$(1.5)\qquad \lambda(\bar{x}) = \sum_{n=1}^{\infty} 2^{-n}x_n.$$

The function λ, which maps G on to the closed interval [0, 1], does not

1) In what follows, the periodicity with period 1 is assumed for any function.

2) we shall abbreviate "Walsh-Fourier series" as WFS and "trigonometric Fourier series" as TFS.