ON ASYMPTOTICALLY ABSOLUTE CONVERGENCE

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Let us consider the series

$$\sum_{n=1}^{\infty} a_n \tag{1}$$

of real numbers a_n. We shall say that the series (1) is *asymptotically absolutely convergent* if there exists an increasing sequence of positive integers $\{n_k\}$ such that $k/n_k \to 1$ as $k \to \infty$ and the subseries

$$\sum_{k=1}^{\infty} a_{n_k} \tag{2}$$

converges absolutely.

We shall establish, in this note, a theorem of Tauberian type and some results for trigonometrical series.

1. Tauberian theorem.

THEOREM 1. *Suppose that the series* (1) *is asymptotically absolutely convergent, and one of the following three conditions is satisfied*:

(i) $\{|a_n|\}$ *is a monotone sequence*;

(ii) $|a_{n+1}| < (1 + C/n)|a_n|$ $(n \geqq n_0)$, *where C and n_0 are positive constants independent of n*;

(iii) *for some B which is independent of $N = 1, 2, \cdots$*,

$$\sum_{n=1}^{N-1} n\Big|\,|a_n| - |a_{n+1}|\,\Big| + N|a_N| \leqq B \sum_{n=1}^{N} |a_n|. \tag{3}$$

Then the series (1) *converges absolutely*.

PROOF. If (i) is satisfied, then the absolute convergence of the series of type (2) implies the decreaseness of $|a_n|$; and (i) is included in (ii). On the other hand, (ii) implies the inequality (3). For, Supposing $n_0 = 1$,

$$\begin{aligned}
&\sum_{n=1}^{N-1} n\Big|\,|a_n| - |a_{n+1}|\,\Big| + N|a_N| \\
&\qquad \leqq \sum_{n=1}^{N-1} n\frac{C}{n}\,|a_n| + \sum_{n=1}^{N}\Big(1 + \frac{C}{N-1}\Big)\Big(1 + \frac{C}{N-2}\Big)\cdots\Big(1 + \frac{C}{n}\Big)\,|a_n| \\
&\qquad \leqq C\sum_{n=1}^{N-1} |a_n| + e^C \sum_{n=1}^{N} |a_n| \leqq (C + e^C)\sum_{n=1}^{N} |a_n|.
\end{aligned}$$

Hence it is sufficient to prove the absolute convergence of (1) under the condition (iii). Suppose that (2) converges absolutely and $k/n_k \to 1$ as $k \to \infty$. Let $\varepsilon_n = 1$ if $n = n_k$, $k = 1, 2, \cdots$, and $\varepsilon_k = 0$ otherwise. Then, as we see easily,