ON THE STRUCTURE OF A SPHERE BUNDLE

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1. In this note, we shall give some properties of a kind of sphere bundles. As for the definition of the fibre bundle, see for example, N. E. Steenrod [3].

THEOREM 1. *Let S^m be an m-dimensional sphere, and M^n be an $(n-m)$-sphere bundle over S^m $(n > m \geqq 2)$. In addition, let the homotopy groups $\pi_i(M^n)$ vanish for $i = 1, \cdots, m$. Then n must be equal to $2m-1$.*

PROOF.[1] In considering the homotopy groups of M^n, let x_0 be their base point, and S_0^{n-m} be the fibre over x_0. Then we get the following exact homotopy sequence

(1.1) $$\cdots\cdot \to \pi_i(M^n) \to \pi_i(M^n, S_0^{n-m}) \to \pi_{i-1}(S_0^{n-m}) \to \pi_{i-1}(M^n) \to \cdots\cdot.$$

According to the hypothesis on $\pi_i(M^n)$ and to the exactness of (1.1), we get

(1.2) $$\pi_i(M^n, S_0^{n-m}) \approx \pi_{i-1}(S_0^{n-m}) \qquad (2 \leqq i \leqq m).$$

On the other hand, as M^n is compact, the covering homotopy theorem holds in this case, and according to the theorem of W. Hurewicz and N. E. Steenrod [2], we get

(1.3) $$\pi_i(M^n, S_0^{n-m}) \approx \pi_i(S^m) \begin{cases} \approx \text{infinite cyclic group, for } i = m, \\ \approx 0 \text{ for } 2 \leqq i \leqq m-1, \text{ if } m \geqq 3. \end{cases}$$

From (1.2) and (1.3), we can easily obtain the required conclusion.

Evidently, this theorem can be extended to the generalized spaces, where the structure of homotopy groups and the dimensionalities are the same as we have quoted above.

2. In this section, we shall consider particularly, the orientable manifold M^{2m-1}, which is an $(m-1)$-sphere bundle over S^m. In addition, we shall assume that the projection $\pi : M^{2m-1} \to S^m$ is algebraically inessential.

Let the oriented sphere S^m be situated in an $(m+1)$-Euclidean space $(x_1, \cdots x_{m+1})$ by the equation

$$x_1^2 + \cdots\cdot + x_{m+1}^2 = 1.$$

And we shall separate S^m into two hemispheres E_1^m, E_2^m respectively,

$$E_1^m = \{(x_i) \in S^m \mid x_{m+1} \geqq 0\}, \quad E_2^m = \{(x_i) \in S^m \mid x_{m+1} \leqq 0\}.$$

We shall define $S_0^{m-1} = E_1^m \frown E_2^m$, $O_1 = (0, \cdots\cdot 0, 1)$, $O_2 = (0, \cdots\cdot, 0, -1)$, and orient S_0^{m-1} coherently with E_1^m and conversely with E_2^m.

It is well known [1], that the fibre bundle over an element must be a product bundle. Therefore, there must exist two homeomorphisms φ_1 and

1) Prof. K. Aoki suggested me the proof of this theorem.