ON THE PRINCIPAL GENUS THEOREM CONCERNING THE ABELIAN EXTENSIONS

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Introduction

The principal genus theorem in a cyclic extension plays an important role in the study of the class field theory. A generalization of this theorem in the case of an abelian extension will be shown in this note. It was a long standing conjecture of Professor Tadao Tannaka.

Let K be an abelian extension of an algebraic number field k, and $\mathfrak{f}$ and $\mathfrak{F}$ be the conductor and the "Geschlechtermodul" of K/k respectively. Let $\mathfrak{m}$ be an integral module in k. Let us denote the ray ("Strahl") mod. $\mathfrak{m}\mathfrak{f}$ in k and mod. $\mathfrak{m}\mathfrak{F}$ in K by $R_k(\mathfrak{m}\mathfrak{f})$ and $R_K(\mathfrak{m}\mathfrak{F})$, respectively. H. Hasse proved the following so-called principal genus theorem for a cyclic extension (Cf. [1], pp. 304–310):

If K/k is a cyclic extension, following two conditions concerning an ideal $\mathfrak{A}$ of K are equivalent:

(1) $$N_{Kk}\,\mathfrak{A} \in R_k(\mathfrak{m}\mathfrak{f}),$$

(2) $$\mathfrak{A} = \mathfrak{B}^{1-\sigma}(A),\ \ (A) \in R_K(\mathfrak{m}\mathfrak{F}),$$

where σ is a generator of Galois group G of K/k.

The generalization in quite the same form seems to be difficult, and we take up the transformation set instead of the norm in (1). Namely, starting from a given ideal $\mathfrak{A}$, define an ideal $\mathfrak{A}(\sigma^a)$ corresponding to each element σ^a of G as the following:

$$\mathfrak{A}(1) = 1, \qquad \mathfrak{A}(\sigma) = \mathfrak{A}, \qquad \mathfrak{A}(\sigma^a) = \mathfrak{A}(\sigma)^{\sigma^{a-1}}\mathfrak{A}(\sigma^{a-1}) \qquad (0 < a < e,\ \ \sigma^e = 1).$$

Then, on the one hand, the condition (1) is equivalent to the condition

(3) $$\mathfrak{A}(\rho)^{\tau}\mathfrak{A}(\tau)\mathfrak{A}(\rho\tau)^{-1} \in R_k(\mathfrak{m}\mathfrak{f})$$

for all ρ, τ in G. And, on the other hand, the condition (2) is equivalent to the existence of an ideal $\mathfrak{B}$ such that

(4) $$\mathfrak{A}(\rho) = \mathfrak{B}^{1-\rho}(A(\rho)),\ \ (A(\rho)) \in R_K(\mathfrak{m}\mathfrak{F})$$

for any ρ in G. Moreover, these numbers $A(\rho)$ satisfy the condition:

(5) $$A(\rho)^{\tau}A(\tau)A(\rho\tau)^{-1} \equiv 1 \ \text{mod. } \mathfrak{m}\mathfrak{f}, \text{ and is contained in } k$$

for any ρ, τ in G. So that, in the case of a cyclic extension, the assertion (1)$\rightarrow$(2) is equivalent to the assertion (3)$\rightarrow$(4), (5).

In an arbitrary abelian extension K/k, we shall deal with a generalization in this form. Let us denote by $\{\mathfrak{A}(\rho)\}$ a system of ideals in K corresponding to the elements of Galois group G of K/k. The main theorem in this note