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This paper consists of two independent parts. The first part concerns uniform convergence of Fourier series and the second gives an approximation formula.

PART I.

1. R. Salem [1] has given a very general test for uniform convergence of Fourier series, which includes known criteria. By his idea, T. Kawata and the author [2] have given a test for uniform Cesàro summability of Fourier series. The last test, as is shown in [2], contains theorems due to Zygmund, Wiener-Marcinkiewicz and Salem. But the theorem due to Hardy-Littlewood [3] (Theorem 2) is not contained. We shall now generalize above criteria such that the last theorem is contained.

2. THEOREM 1. *If $f(x)$ is a continuous function such that*

$$(1)\qquad \sum_{k=1}^{[n/2]} \frac{n}{k^{1+\alpha}} \int_{\pi/n}^{2\pi/n} |f(x+t+2k\pi/n) - f(x+t+(2k+1)\pi/n)|\,dt = o(1)$$

uniformly in x for $-1 < \alpha < 1$, then the Fourier series of $f(x)$ is summable (C, α) uniformly.

PROOF. Let $\sigma_n^\alpha(x)$ be the n-th Cesàro mean of the Fourier series of $f(x)$ of order α. Then

$$\delta_n(x) = \sigma_n^\alpha(x) - f(x) = \frac{1}{\pi}\int_0^\pi \varphi_x(t) K_n^\alpha(t)\,dt$$

where $K_n^\alpha(t)$ is the Fejér kernel of order α. It is well known that

$$K_n^\alpha(t) = \psi_n^\alpha(t) + r_n^\alpha(t)$$

where

$$(2)\qquad \psi_n^\alpha(t) = \cos\left(\left(n + \frac{1+\alpha}{2}\right)t - \frac{1-\alpha}{2}\pi\right)\Big/ A_n^\alpha\left(2\sin\frac{t}{2}\right)^{1+\alpha},$$

$$(3)\qquad r_n^\alpha(t) = O(1/nt^2).$$

We have

$$\delta_n(x) = \frac{1}{\pi}\int_0^\pi = \frac{1}{\pi}\int_0^{\pi/n} + \frac{1}{\pi}\int_{\pi/n}^\pi = I_1 + I_2.$$

By the continuity of $f(x)$, $I_1 = o(1)$ uniformly. Concerning I_2, we have

$$I_2 = \frac{1}{\pi}\int_{\pi/n}^\pi \varphi_x(t)\psi_n^\alpha(t)\,dt + \frac{1}{\pi}\int_{\pi/n}^\pi \varphi_x(t) r_n^\alpha(t)\,dt = I_3 + I_4,$$