ON THE STRONG LAW OF LARGE NUMBERS

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1. The purpose of this paper is to state an extension of Kolmogorov's theorem [3] which provides a necessary and sufficient condition for the validity of the strong law of large numbers for a sequence of independent, identically distributed random variables.

We consider the probability space $(\mathbf{X}, P)$ such that $\mathbf{X}$ is a space whose points are denoted by t and P is a probability measure. Then our extension is stated as follows.

THEOREM. *Let $\{X_n(t)\}$ be a sequence of independent random variables satisfying that*
(1.1) *there exists a positive constant K such that, for any positive integer m and for any extended real numbers*[1] $a_1, b_1, \ldots, a_m, b_m$,

$$\limsup_n \frac{1}{n} \sum_{i=0}^{n-1} P\{t\,;\ a_1 \leqq X_{1+i}(t) < b_1, \ldots, \ a_m \leqq X_{m+i}(t) < b_m\}$$
$$\leqq K \cdot P\{t; a_1 \leqq X_1(t) < b_1, \ldots, \ a_m \leqq X_m(t) < b_m\}.$$

Then the following (1.2) *and* (1.3) *are equivalent.*

$$\sum_{n=1}^{\infty} \int_{|X_n(t)| \in A_n} |X_n(t)|\, dP < \infty \tag{1.2}$$

for some Borel sets $A_1, A_2, \ldots$ satisfying

$$A_i \cap A_j = 0 \quad (i \neq j), \qquad \bigcup_{n=1}^{\infty} A_n = [0, \infty),$$

where some of A_n's may be empty.

$$P\left\{ t\,;\ \lim_n \frac{1}{n} \sum_{i=1}^{n} X_i(t) = c \right\} = 1 \tag{1.3}$$

for some constant c.

The proof appears in § 2.

If $\{X_n(t)\}$ is identically distributed, (1.1) holds trivially and the sum in (1.2) is equal to the first absolute moment $E(|X_1|)$ common for all X_n's, so that the theorem is reduced to Kolmogorov's.

2. To prove the theorem we need a lemma. Before stating this we must prepare several definitions and notations.

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1) By an extended real number we mean either a usual real number or one of the symbols $+\infty$ and $-\infty$. In what follows we make the convention that when $a = -\infty$, "$a \leqq$" is replaced by "$a <$".