ON A THEOREM OF LINDELÖF CONCERNING PRIME ENDS

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A short proof of the following well known theorem of Lindelöf has been given by Tsuji [2].

THEOREM. *Let D be a bounded simply-connected domain, and let $w = f(z)$ map D conformally on $|w| < 1$. If $\{z_n\}$ is a sequence of points of D such that the sequence $w_n = f(z_n)$ converges to a point α of $|w| = 1$ in a Stolz angle, $|\arg(\alpha - w)| < \frac{1}{2}\pi - \delta$, then every limit-point of the sequence $\{z_n\}$ is a principal point*[1)] *of the prime end of D which corresponds to α.*

In this note we show how the proof of the theorem may be simplified still further by using a very elementary topological argument.

Tsuji proves the theorem by combining the following results.

LEMMA A. *Let Dba bounded simply-connected domain, and let $w = f(z)$ map D conformally on $|w| < 1$. Let $\{\rho_n\}$ be a sequence of positive numbers such that $\rho_{n+1} \leqq \frac{1}{2}\rho_n < 1$, and let S_n be the domain $\frac{1}{2}\rho_n < |1 - w| < \rho_n$, $|w| < 1$. Then we can find an increasing sequence $\{n_\nu\}$ of positive integers, and a chain $\{q_\nu\}$ of cross-cuts of D associated with the prime end of D which corresponds to $w = 1$, such that the image of q_ν in $|w| < 1$ is an arc of a circle with centre $w = 1$ lying in S_{n_ν}.*

LEMMA B. *Let D be a bounded simply-connected domain, $F(D)$ its frontier, let $w = f(z)$ map D conformally on $|w| < 1$, and let $z = \psi(w)$ be the inverse of f. Let $\{\rho_n\}$ be a sequence of positive numbers such that $\rho_{n+1} \leqq \frac{1}{2}\rho_n < 1$, and let T_n be the domain $\frac{1}{2}\rho_n < |w - 1| < \rho_n$, $|w| < 1$, $|\arg(1 - w)| < \frac{1}{2}\pi - \delta < \frac{1}{2}\pi$. Then we can find an increasing sequence $\{n_\nu\}$ of positive integers such that the values of $\psi(w)$ in $\overline{T}_{n_\nu}$ converge to a point a of $F(D)$.*

The deduction of the theorem from these two lemmas is immediate. For we may suppose that $\alpha = 1$, and that $\{z_n\}$ converges to a point a of $F(D)$. We can then find a chain of cross-cuts $\{q_\nu\}$, associated with the prime end of D which corresponds to $w = 1$, converging to the point a, and this is the required result.

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1) For the definition of this and other terms belonging to the theory of prime ends, see Carathéodory [1].