ON SOME RANDOM RIEMANN-SUMS

SHIGERU TAKAHASHI

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1. In the present note $\{t_i(\omega)\}$, $i = 1, 2, \ldots,$ will denote a sequence of independent random variables defined on a probability space $(\Omega, \mathbf{B}, P)$ and each $t_i(\omega)$ have the uniform distribution on the interval $[0, 1]$, that is, for $0 \leqq x \leqq 1$

$$P[t_i(\omega) < x] = x.$$ [1)]

For each ω let $t_{i,n}(\omega)$ be the i-th value of $\{t_j(\omega)\}$ $(1 \leqq j \leqq n)$ arranged in the increasing order and let, for all n,

$$t_{0,n}(\omega) \equiv 0 \quad \text{and} \quad t_{n+1,n}(\omega) \equiv 1.$$

Further let $f(t)$, $0 \leqq t \leqq 1$, denote a Borel-measurable and integrable function.

It is an interesting problem, proposed by K. Ito, whether the following Riemann-sums

$$S_n(\omega) = \sum_{i=1}^{n} f(t_{i,n}(\omega))(t_{i+1,n}(\omega) - t_{i,n}(\omega)) \tag{1.1}$$

converge to $\int_0^1 f(t)\,dt$ or not, in any sense. In [2] we proved that under certain local conditions, we have

$$P\left[\lim_{n\to\infty} S_n(\omega) = \int_0^1 f(t)\,dt\right] = 1. \tag{1.2}$$

In this note we prove the following

THEOREM 1. *If $f(t) \in L_p(0, 1)$ $p > 1$, then* (1.2) *holds.*

For $f(t) \in L(0, 1)$ we can not prove whether (1.2) holds or not.

2. Let us put, for $1 \leqq i \leqq n$ and $n = 1, 2, \ldots,$

$$d_{i,n}(\omega) = t_{j+1,n}(\omega) - t_i(\omega), \ \text{if } t_i(\omega) = t_{j,n}(\omega) \qquad (j = 1, 2, \ldots, n) \tag{2.1}$$

and

$$d'_{i,n}(\omega) = t_i(\omega) - t_{j-1,n}(\omega), \ \text{if } t_i(\omega) = t_{j,n}(\omega) \qquad (j = 1, 2, \ldots, n). \tag{2.1'}$$

Then we can write

$$S_n(\omega) = \sum_{i=1}^{n} d_{i,n}(\omega) f(t_i(\omega)) \tag{2.2}$$

and

1) For the notations and definitions in the theory of probability see [1].