SIMILARITIES AND DIFFERENTIABILITY

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1. Introduction. Prof. K. Yano mentioned in a lecture that a Riemann space is euclidean if it possesses a one-parameter group H of non-isometric similarities. With "Minkowskian" replacing "euclidean" the theorem holds also for Finsler spaces, but fails to hold when differentiability hypotheses are altogether omitted by substituting G-spaces[1] for Finsler spaces. This remains so, even under very strong supplementary hypotheses: *without being Minkowskian the space may, in addition to H, possess groups of motions of a rather high dimension and its geodesics may be the euclidean straight lines.* On the other hand, a very mild differentiability suffices for concluding from the existence of H and the axioms of a G-space that the space is Minkowskian. Yet, nothing in the formulation of the original theorem suggests the necessity of smoothness requirements.

The author is not aware of any similarly striking example where differentiability assumptions in their usual form conceal strong purely geometric implications. Therefore a systematic analysis of the situation seems justified,

We begin by discussing similarities in general G-spaces, then convince ourselves by examples[2] that the above mentioned phenomena actually occur. Next, we discuss a simple intrinsic, geometric condition for differentiability. Examples show that this condition is still too weak to deduce the Minkowskian character of the metric from the existence of H, because, in fact, the local metric need not be Minkowskian.

However, strengthening the condition slightly into an analogue of continuous differentiability proves sufficient: *A G-space which admits a similarity with dilation factor $k \neq 1$ (a group H of similarities is not needed) and is continuously differentiable at one of the (always existing) fixed points of the similarity, is Minkowskian in the small when $k > 1$, and in the large when $k < 1$.*

The local metric is Minkowskian at a point of a G-space where the space is continuously differentiable and regular[3]. We use these methods to partially solve the interesting problem of *deciding from the intrinsic*

1) G-spaces are defined in [2, page 37]; although they have no differentiability properties, a large part of differential geometry holds for them, see [2].

2) A cone in E^3 with total angle $\alpha<2\pi$ or $\alpha>2\pi$ at its apex a and with its intrinsic metric provides a simple example for a space which possesses a group H of similarities (and the group of rotations about a). However, for $\alpha<2\pi$ prolongation of a segment for a point b to the apex a beyond a is impossible, and for $\alpha>2\pi$ it is not unique, so that the axioms for a G-space are not satisfied.

3) An other approach to differentiability of G-spaces is found in [1, Chapter II]. The present conditions are simpler and the proofs shorter. Because the author suspected the existence of such an approach, the method of [1, Chapter II] is not discussed in [2]. The remaining results of [1] are also found in [2].