ON GENERAL ERGODIC THEOREMS II

SHIGERU TSURUMI

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1. Introduction. E. Hopf [6] has established a pointwise ergodic theorem which asserts the convergence almost everywhere of averages $\frac{1}{n}\sum_{j=0}^{n-1} T^j f$ where T is an operator defined by a Markov process with an invariant distribution and where f is an integrable function. Recently this theorem has been extended to one for more general operator by N. Dunford and J. T. Schwartz [4]. We shall here observe the convergence almost everywhere of averages $\sum_{j=0}^{n-1} T^j f \Big/ \sum_{j=0}^{n-1} T^j g$ where T is a linear positive operator with some restrictions and where f and g are integrable and g is positive almost everywhere.

2. Notations and preliminaries. Let $(X, \mathfrak{F}, \mu)$ be a finite measure space such that X is a set and $\mathfrak{F}$ a σ-field consisting of subsets of X and μ a non-negative countably additive set function defined on $\mathfrak{F}$ and $\mu(X) < +\infty$.

Throughout this paper, "measurable", "almost all (almost everywhere)" and "integrable" mean "$\mathfrak{F}$-measurable", "μ-almost all (μ-almost everywhere)" and "μ-integrable", respectively, and every function under consideration is real-valued.

We denote by $L_1(A)$ the Lebesgue space of measurable integrable functions f defined on $A \in \mathfrak{F}$, the norm being

$$|f|_1 = \int_A |f(x)|\mu(dx),$$

and by $L_\infty(A)$ the Lebesgue space of measurable essentially bounded functions f defined on $A \in \mathfrak{F}$, the norm being

$$|f|_\infty = \operatorname*{ess\,sup}_{x \in A} |f(x)|.$$

If $A = X$, we drop "X" in $L_1(X)$ and $L_\infty(X)$ and write L_1 and L_∞.

Let f and g be measurable and $A \in \mathfrak{F}$. If $f(x) \geqq g(x)$ for almost all $x \in A$, we write "$f \geqq g$ in A". Further, "$f > g$ in A" and "$f = g$ in A" are defined in like manner. If $A = X$, we drop the term "in X".

Let T be a linear operator of L_p into itself where $p = 1$ or ∞. If T is a continuous operator, the operator norm of T is defined as usual and denoted by $|T|_p$. The operator T is called positive provided that $Tf \geqq 0$ for every $f \in L_p$ with $f \geqq 0$. A set $A \in \mathfrak{F}$ is called T-invariant provided that

$$T(f \cdot e_A) = Tf \quad \text{in } A$$