ON A CONJECTURE OF KAPLANSKY

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Prof. Kaplansky stated a conjecture that any derivation of a C^*-algebra would be automatically continuous [1]. In this note, we shall show that this conjecture is in fact true.

THEOREM. *Any derivation of a C^*-algebra is automatically continuous.*

PROOF. Let A be a C^*-algebra, $'$ a derivation of A. It is enough to show that the derivation is continuous on the self-adjoint portion A_s of A. Therefore if it is not continuous, by the closed graph theorem there is a sequence $\{x_n\}$ $(x_n \neq 0)$ in A_s such that $x_n \to 0$ and $x_n' \to a + ib(\neq 0)$, where a and b are self-adjoint. First, suppose that $a \neq 0$ and there exists a positive number $\lambda(> 0)$ in the spectrum of a (otherwise consider $\{-x_n\}$). It is enough to assume that $\lambda = 1$.

Then there is a positive element h $(\|h\| = 1)$ of A such that $hah \geqq \frac{1}{2}h^2$. Put $y_n = x_n + 3 \cdot \|x_n\| \cdot I$, then $y_n \to 0$, $y_n' = x_n'$ and $(hy_nh)' = h'y_nh + hy_n'h + hy_nh'$; hence $(hy_nh)' \to h(a + ib)h$.

Therefore

$$\|(hy_{n_0}h)' - h(a + ib)h\| < \frac{1}{8} \qquad \text{for some } n_0 \quad \cdots\cdots\cdots\cdots(1).$$

On the other hand

$$hy_nh \leqq 4\|x_n\|h^2 \text{ and } \frac{1}{2} \cdot \frac{hy_nh}{4\|x_n\|} \leqq hah \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Since $\|x_n\| \cdot I + x_n \geqq 0$, $\dfrac{hy_nh}{4\|x_n\|} \geqq \dfrac{1}{2}h^2$.

Hence

$$\left\| \frac{hy_nh}{4\|x_n\|} \right\| \geqq \frac{1}{2}\|h\|^2 = \frac{1}{2} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3)$$

Let C be a C^*-subalgebra of A generated by $hy_{n_0}h$ and I, then by the (3) there is a character φ of C such that $\varphi\left(\dfrac{hy_{n_0}h}{4\|x_{n_0}\|}\right) \geqq \dfrac{1}{2}$.