PROJECTIVE MODULES OVER SEMILOCAL RINGS

YUKITOSHI HINOHARA

(Received March 31, 1962)

Let R be a commutative ring with a unit element. If there exist no proper ideals $\mathfrak{a}$, $\mathfrak{b}$ such that $R = \mathfrak{a} \oplus \mathfrak{b}$, then R is said to be indecomposable. If the number of maximal ideals of R is finite, then R is said to be semilocal. In [6], I. Kaplansky proved that, over a local ring, any projective module is free. Our objective in this paper is to generalize his theorem into

THEOREM. *Over a commutative indecomposable semilocal ring, any projective module is free.*

Every ring considered in this paper has a unit element which acts as unit operator on any module. Λ denotes a ring (not always commutative) and R denotes a commutative ring. Modules are always left modules.

1. Some lemmas on projective modules.

We begin with a trivial

LEMMA 1. *Let L, M, N be modules over a ring Λ such that $L \supset M \supset N$. If N is a direct summand of L, N is a direct summand of M.*

PROOF. Let $L = N \oplus N'$. Then we have $M = N \oplus (N' \cap M)$.

LEMMA 2. *Let P be a projective module over a ring Λ and p an element of P. If $p \notin \mathfrak{m} P$ for any maximal right ideal $\mathfrak{m}$ of Λ, then Λp is a direct summand of P and p is a free basis of Λp, where $\mathfrak{m}P$ is the image of $\mathfrak{m} \otimes_{\Lambda} P \to P$ by the natural map.*

PROOF. Let F be a free module such that $F = P \oplus Q$, $\{u_i\}$ a basis of F;

$$p = \sum_{i=1}^{n} r_i u_i, \ r_i \in R;$$

$$u_i = p_i + q_i, \quad p_i \in P, \ q_i \in Q.$$

Then we have that the right ideal $(r_1, \ldots\ldots, r_n)$ generated by r_i is equal to Λ, since, if $r_i \in \mathfrak{m}$ for a maximal right ideal $\mathfrak{m}$, we have

$$p = \sum r_i p_i,$$

i. e., $p \in \mathfrak{m} P$. Therefore there exist elements $s_1, \ldots\ldots, s_n$ in Λ such that