APPROXIMATION OF FUNCTIONS BY RIESZ MEAN OF THEIR FOURIER SERIES

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Let $\varphi(u)$ be defined in $0 \leqq u \leqq 1$ and continuous at $u = 0$ and of bounded variation on (0, 1). Then we consider the mean of series $\sum a_n$ by

$$\sum_{k=0}^{n-1} \varphi\left(\frac{k}{n}\right) a_k, \quad \varphi(0) = 1.$$

When $\varphi(u) = (1 - u^\beta)^\delta$ $(\beta, \delta > 0)$, we say this Riesz mean of the series.

Let $f(x)$ be periodic and integrable over $(0, 2\pi)$, let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(x),$$

and be its Riesz mean

$$R_n(x,f) = A_0 + \varphi\left(\frac{1}{n}\right) A_1(x) + \cdots + \varphi\left(\frac{n-1}{n}\right) A_{n-1}(x), \ \varphi(u) = (1 - u^\beta)^\delta.$$

When $\delta = 1$ and β is an integer, the approximation of $f(x)$ by Riesz mean $R_n(x,f)$ was solved by Zygmund [4].

Sz. Nagy [3] treated the general case. He did not calculate completely, but if we calculate following his method, we have,

THEOREM A. (SZ. NAGY). *If* $f(x)$ *is r-times differentiable and* $f^{(r)}(x) \in \mathrm{Lip}\ \alpha\ (0 < \alpha \leqq 1)$, *then*

$$|R_n(x,f) - f(x)| = O\left(\frac{1}{n^{\alpha+r}}\right), \ \text{if } \gamma > \alpha + r,$$

$$|R_n(x,f) - f(x)| = O\left(\frac{\log n}{n^{\alpha+r}}\right), \ \text{if } \gamma = \alpha + r, (*)$$

$$|R_n(x,f) - f(x)| = O\left(\frac{1}{n^{\gamma}}\right), \ \text{if } \gamma < \alpha + r,$$

where $\gamma = \min(\beta, r + \delta)$. *In the special case* $\alpha = 0$ *and* $r =$ *an even integer of* (*), *the factor* log n *is suppressed.*

From this, we may infer that his order of approximation depends upon δ.