ON BOREL SERIES

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Introduction. In his book "Leçons sur la théorie des fonctions", É.Borel [1] considers certain infinite series, which in the theory of real variables, are sometimes referred to as Borel series, and which are defined by

$$(1) \qquad \sum_{n=1}^{\infty} (A_n/r_n^{m_n}),$$

$r_n^2 = (x_1 - a_n^{(1)})^2 + (x_2 - a_n^{(2)})^2 + \cdots + (x_h - a_n^{(h)})^2$, $m_n < m$. A_n, $n = 1, 2, \cdots$, is a sequence of real numbers and $\sum A_n$ is assumed to be convergent. $a_n^{(h)}$, $n = 1,2, \cdots$ are $h \geqq 1$ sequences of real numbers; $x_1, \cdots, x_h$ are real variables and the exponents m_n are real positive numbers. According to a theorem of Borel, (1) converges almost everywhere.

If we put $A_n \equiv |A|^{p+q}/q$, where $|A_0| < |A| < 1$, A not necessarily real, p and q positive integers, and $m_n = 1$, $r_n = |x_1 - a_n^{(1)}| \equiv |x - (p/q)|$, i. e. if we identify $\{a_n^{(1)}\}$ with the somehow simply ordered double sequence $\{\{(p/q)\}\}$, then we obtain the special case of (1)

$$(2) \qquad f(A\,; x) \equiv \sum_{p,\,q=1}^{\infty} (|A|^{p+q}/|qx - p|).$$

In (2), let x be a fixed, real irrational number. Historically (2) was first discussed by H.Bruns [2], while dealing with the convergence of a trigonometric series to a bounded function for certain values of the parameters which occur. Since then, (2) has received considerable attention, e.g. [3], [4].

The principal purpose of this paper is to discuss convergence and divergence of (2) from an arithmetic point of view and in particular show that (2) may still converge for a subset of Liouville numbers x. The main tool will be K.Mahler's [5] classification of numbers as it is exposed in detail by Th.Schneider [6]. It would of course be nice to obtain an "if and only if" theorem to the effect that (2) diverges for all real irrational x that satisfy certain properties. Unfortunately, this appears unattainable at this time, owing in part to the method applied.

In the first paragraph, in which we shall point out a generalization of a theorem of E. Maillet, we shall find some aspects which are relevant to some

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