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ON THE SOLUTION OF A FUNCTIONAL INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper the solutions of the functional inequality,

$$\sum_{i=1}^{n} g(p_i)f(p_i) \geqq \sum_{i=1}^{n} g(p_i)f(q_i),\ p_i > 0,\ q_i > 0,\ \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$$

for $n \geqq 2$ for given g and under various assumptions on f are obtained. Some information-theoretic applications of the functional inequality are also given.

1. **Introduction.** Consider two probablity distributions $P=(p_1, \cdots, p_n)$ and $Q=(q_1, \cdots, q_n)$ where $p_i > 0$, $q_i > 0$ for all $i=1, \cdots, n$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$. Then this paper deals with the solutions of the functional inequality,

$$\sum_{i=1}^{n} g(p_i)f(p_i) \geqq \sum_{i=1}^{n} g(p_i)f(q_i), \tag{1.1}$$

for $n \geqq 2$ and for given g and under various assumptions on f. Three theorems are proved. The first theorem gives the solutions of (1.1) for $n > 2$, $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, where it is assumed that g is positive, strictly monotonic increasing and continuous in $(0, 1)$ and no regularity assumption is made on f. The second and third theorems deal with the differentiable and absolutely continuous solutions of

$$g(p)f(p) + g(1-p)f(1-p) \geqq g(p)f(q) + g(1-p)f(1-q),\ p,\ q \in (0,1)\,; \tag{1.2}$$

where g is supposed to be positive and strictly monotonic increasing and positive, strictly monotonic increasing and continuous respectively.

Some interesting special cases indicating applications to Information Theory are also given towards the end of the paper.

2. **Solution of** (1.1). In this section we prove the following theorem:

THEOREM 1. *Every solution of the inequality,*