Tôhoku Math. Journ.
25 (1973), 33-52.

EULER-POINCARÉ CHARACTERISTICS AND CURVATURE TENSORS

SHÛKICHI TANNO*)

(Received June 15, 1972)

1. Introduction. Let (M, g) be a Riemannian manifold with positive definite metric tensor $g = (g_{ij})$. By $R = (R^i{}_{jkl})$, $R_1 = (R_{jk})$ and S we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature, respectively. The dimension of M is denoted by m. We denote $(R, R) = R_{ijkl}R^{ijkl}$ and $(R_1, R_1) = R_{jk}R^{jk}$. Some significance of (R, R), (R_1, R_1) and S is explained in [6] by M. Berger or in [7] by M. Berger-P. Gauduchon-E. Mazet in connection with the Gauss-Bonnet theorem or the spectre of Riemannian manifolds.

We define $A(g)$ and $B(g)$ by

$$A(g) = (R, R) - \frac{2}{m-1}(R_1, R_1)\ , \tag{1.1}$$

$$B(g) = (R_1, R_1) - \frac{1}{m}S^2\ . \tag{1.2}$$

Then we have $A(g) \geqq 0$, and the equality holds on M, $m \geqq 3$, if and only if (M, g) is of constant curvature. $B(g) \geqq 0$ holds, and the equality holds on M, if and only if (M, g) is an Einstein space.

For $m = 2$, $A(g) = B(g) = 0$. (cf. (2.10))

For $m = 3$, $A(g) = 3B(g)$. (cf. (8.3))

For $m \geqq 3$, the best inequality is

$$A(g) - \frac{2m\beta}{(m-1)(m-2)}B(g) \geqq 0\ , \tag{1.3}$$

where β is a real number; $-\infty < \beta < 1$ (cf. Theorem 5.7). The equality holds (at x) if and only if (M, g) is of constant curvature (at x).

After some preliminaries in §2, we study relations among $A(g)$, $B(g)$, Euler-Poincaré characteristic $\chi(M)$, curvature and curvature tensors, Betti numbers, and real homology spheres.

THEOREM A. *Let (M, g) be a compact orientable Riemannian manifold, $m \geqq 3$. Assume one of the followings:*

(a) *(M, g) has positive scalar curvature S and satisfies*

*) The author is partially supported by the Matsunaga Foundation.