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RELATIVE BOUNDEDNESS AND SECOND ORDER DIFFERENTIAL EQUATIONS

DAVID LOWELL LOVELADY

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Abstract. Given two positive continuous functions α and β, necessary and sufficient conditions are given for the system $u''(t) = f(t) + A(t)u(t)$ to have an α-bounded solution u for each β-bounded forcing function f. Applications are given to a nonlinear perturbation problem: $u''(t) = A(t)u(t) + F(t, u(t))$. Indications are given on how to extend these ideas to n^{th} order equations.

I. Introduction. Let Y be a finite-dimensional normed linear space with norm $|\ |$, let R^+ be the set of all nonnegative real numbers, and let $\mathscr{A}[Y]$ be the algebra of linear functions from Y to Y, with induced norm $\|\ \|$. Let A be a continuous function from R^+ to $\mathscr{A}[Y]$. In [2] and [3, Chapter V], W. A. Coppel has obtained necessary and sufficient conditions for it to be true that if f is a bounded continuous function from R^+ to Y then there is a bounded solution u on R^+ of

$$u'(t) = f(t) + A(t)u(t) \,. \tag{1}$$

Coppel's ideas have been amplified and extended by several authors, usually in the direction of determining conditions which ensure that if f is in one of two given function spaces then there is a solution u of (1) in the other. For some recent results in this connection and an excellent discussion of this problem, we refer the reader to T. G. Hallam [4].

In the present work we shall conduct the same kind of study for the second-order problem

$$u''(t) = f(t) + A(t)u(t) \,. \tag{2}$$

If one rewrites (2) as a first-order equation over Y^2 and then invokes known results, one's hypotheses require extending the class of forcing functions in a way unnatural to our purposes, and one's conclusions give boundedness properties not only for u but also for u'. (Compare the discussion of J. L. Massera and J. J. Schäffer [6, Chapter 12, § 120].) Thus we see the rationale for studying (2) as is.

II. Relatively bounded solutions. If γ is a positive continuous func-