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A CHARACTERIZATION OF METRIC SPHERES IN HYPERBOLIC SPACE BY MORSE THEORY

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0. Introduction. Let M^n be a differentiable manifold of class C^∞. By a Morse function f on M^n, we mean a differentiable function f on M^n having only non-degenerate critical points. A well-known topological result of Reeb states that if M^n is compact and there is a Morse function f on M^n having exactly 2 critical points, then M^n is homeomorphic to an n-sphere, S^n (see, for example, [3], p. 25).

In a recent paper, [4], Nomizu and Rodriguez found a geometric characterization of a Euclidean n-sphere $S^n \subset R^{n+p}$ in terms of the critical point behavior of a certain class of functions L_p, $p \in R^{n+p}$, on M^n. In that case, if $p \in R^{n+p}$, $x \in M^n$, then $L_p(x) = (d(x, p))^2$, where d is the Euclidean distance function.

Nomizu and Rodriguez proved that if M^n $(n \geqq 2)$ is a connected, complete Riemannian manifold isometrically immersed in R^{n+p} such that every Morse function of the form L_p, $p \in R^{n+p}$, has index 0 or n at any of its critical points, then M^n is embedded as a Euclidean subspace, R^n, or a Euclidean n-sphere, S^n. This result includes the following: if M^n is compact such that every Morse function of the form L_p has exactly 2 critical points, then $M^n = S^n$.

In this paper, we prove results analogous to those of Nomizu and Rodriguez for a submanifold M^n of hyperbolic space, H^{n+p}, the space-form of constant sectional curvature -1.

For $p \in H^{n+p}$, $x \in M^n$, we define the function $L_p(x)$ to be the distance in H^{n+p} from p to x. We then define the concept of a focal point of (M^n, x) and prove an Index Theorem for L_p which states that the index of L_p at a non-degenerate critical point x is equal to the number of focal points of (M^n, x) on the geodesic in H^{n+p} from x to p.

In section 2, we prove that a metric sphere $S^n \subset H^{n+p}$ can be characterized by the condition that every Morse function of the form L_p, $p \in H^{n+p}$, has exactly 2 critical points.

In section 3, we give an example which shows that a result analo-

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