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δ-COMMUTING MAPPINGS AND BETTI NUMBERS

Dedicated to Professor Carl B. Allendoerfer, 1911-1974.

BILL WATSON

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The Hodge-de Rham theorem [3] for oriented, compact, Riemannian manifolds says that the classical cohomology groups with real coefficients can be calculated from a knowledge of the linearly independent harmonic differential forms on the manifold. Specifically, let $\mathscr{H}^p(M)$ denote the space of harmonic p-forms on the compact, oriented Riemannian manifold M, and let $H^p(M, R)$ denote the p-th Čech cohomology group with real coefficients. Let $H_d^p(M, R)$ be the de Rham cohomology space; i.e., the quotient vector space,

$$H_d^p(M, R) = \{\mathrm{Ker}\, d \colon \Lambda^p \longrightarrow \Lambda^{p+1}\}/\{\mathrm{Im}\, d \colon \Lambda^{p-1} \longrightarrow \Lambda^p\} .$$

THEOREM (Hodge-de Rham).
(a) *The dimension of* $\mathscr{H}^p(M)$ *is finite, and,*
(b) $H^p(M, R) \cong \mathscr{H}^p(M) \cong H_d^p(M, R)$.

On our compact M, it is easy to show that a harmonic form is in the kernels of both the differential operator d and the codifferential operator δ, simultaneously. Therefore,

$$\mathscr{H}^p(M) = \{\mathrm{Ker}\, d \colon \Lambda^p \longrightarrow \Lambda^{p+1}\} \cap \{\mathrm{Ker}\, \delta \colon \Lambda^p \longrightarrow \Lambda^{p-1}\}$$

and, since we know that any manifold map $\varphi\colon M \longrightarrow N$ onto another compact, oriented, Riemannian manifold, N, commutes with d on the p-forms of N ($\varphi^* d_N = d_M \varphi^*$), it is natural to ask which manifold maps will commute with the codifferential. The hope is that we may find a way to transfer information about $\mathscr{H}^p(N)$ over to $\mathscr{H}^p(M)$ via φ^*, and, thereby, relate their cohomology groups.

We report here the complete classification of all C^2 manifold mappings $\varphi\colon M \longrightarrow N$ between compact, connected, oriented, Riemannian manifolds which satisfy

$$(1) \qquad \varphi^* \delta_N = \delta_M \varphi^*$$

on all of the p-forms of N for a *fixed* $p \geqq 1$. In the case of 1-forms, we find equation (1) to be solved by a rather general class of mappings—