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KINEMATICS AND DIFFERENTIAL GEOMETRY OF SUBMANIFOLDS

——Rolling a ball with a prescribed locus of contact——

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The simplest and most illustrative of the kinematic models we discuss in this paper is the rolling of a ball on its tangent plane. Suppose a smooth curve x_t is given on the unit sphere S^2 (boundary of the unit ball B). Is it possible to roll (without skidding or spinning) the ball B on the tangent plane Σ to S^2 at x_0 in such a way that at each time instant t the point x_t becomes a point of contact with the plane Σ? We shall show that this is possible and that the locus y_t of points of contact on Σ is indeed the development of the curve x_t in the sense of E. Cartan.

When we replace S^2 by an arbitrary smooth surface M, the rolling of M on its tangent plane gives rise to a kinematic interpretation of the Levi-Civita connection for M. We also find that we must impose a certain condition on the curve x_t to prevent the rolling from degenerating into an instantaneous standstill at any instant. This condition is that the tangent vector of x_t is not a principal direction for the zero principal curvature; this condition is satisfied if the curve x_t does not go through a flat point.

In the end we shall study the model of rolling an n-dimensional submanifold M on another n-dimensional submanifold N in a Euclidean space E^m and obtain a kinematic interpretation of the second fundamental form and the normal connection of a submanifold.

The paper is organized as follows. Section 1 is devoted to the basic concepts in kinematics we need. We define the notion of rolling (without skidding or spinning). In Section 2 we discuss the model of rolling a ball and extend it to higher dimensions in Section 3. In Section 4 we treat the rolling of an arbitrary surface on a plane. Section 5 deals with rolling of a surface on another surface. Finally, in Section 6, we discuss the most general question—rolling an n-dimensional submanifold

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