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# NORMAL DERIVATIONS IN OPERATOR ALGEBRAS

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Let $A$ be a ring with involution (briefly, a $*$-ring), $\delta: A \to A$ a derivation of $A$, that is, $\delta(x + y) = \delta x + \delta y$ and $\delta(xy) = (\delta x)y + x(\delta y)$ for all $x, y$ in $A$. When $A$ is a $*$-algebra (over the complex field) one requires also that $\delta$ be a linear mapping. For each $a \in A$ we write $\delta_a$ for the inner derivation implemented by $a$: $\delta_a x = [a, x] = ax - xa$. For a derivation $\delta$ of $A$, the adjoint $\delta^*$ of $\delta$ is the derivation of $A$ defined by the formula $\delta^* x = -(\delta(x^*))^*$; the purpose of the minus sign is to validate the formula $(\delta_a)^* = \delta_{a^*}$. Note also that $\ker(\delta^*) = (\ker \delta)^*$.

In the first part of the paper we explore the relationships between several plausible definitions of normality for a derivation $\delta$ of a $*$-ring, with particular attention to $C^*$-algebras and von Neumann algebras; in the second part, we discuss derivations of certain algebras of "unbounded operators" affiliated with $AW^*$-algebras.

Here are some natural candidates for the definition of "normal derivation" (there seems to be no compelling reason for making a definitive choice):

($N_1$) $\ker \delta = \ker(\delta^*)$;

($N_2$) $\delta^* \delta = \delta \delta^*$;

($N_3$) there exist a $*$-ring $B$ containing $A$ as a $*$-subring, and an element $b \in B$, such that $b$ is normal ($b^* b = b b^*$) and $\delta = \delta_b | A$ (that is, $\delta x = [b, x]$ for all $x \in A$);

($N_4$) there exist a $*$-ring $B$ containing $A$ as a $*$-subring, and an element $b \in B$, such that $\delta = \delta_b | A$ and $A \cap \{b\}'$ is a $*$-subring of $A$, where $\{b\}'$ denotes the commutant of $b$ in $B$;

($N_5$) there exist a $*$-ring $B$ containing $A$ as a $*$-subring, and an element $b \in B$, such that $\delta = \delta_b | A$ and $\{b\}'$ is a $*$-subring of $B$.

The most natural condition is ($N_2$), which mimics the definition of normality for an element of a $*$-ring. The remaining conditions are motivated by the well-known theorem of B. Fuglede: if $\delta = \delta_a$, $a \in A$, then ($N_1$) means that $xa = ax$ if and only if $xa^* = a^* x$; thus, when $A$ is a $*$-algebra of operators in a Hilbert space, $\delta_a$ satisfies ($N_1$) if and only if $a$ is normal (Fuglede's theorem [13, Prob. 152]). Here are some elementary relations between these conditions: