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RANKIN'S METHOD IN THE CASE OF LEVEL $4q$ AND ITS APPLICATION TO THE DOI-NAGANUMA LIFTING

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Introduction. Let $S_k(\Gamma_0(N), \chi)$ be the space of integral cusp forms of Neben-type χ and of weight k with respect to $\Gamma_0(N)$. We associate with a cusp p of $\Gamma_0(N)$ a matrix $\alpha_p(\in SL_2(\boldsymbol{R}))$ such that $\alpha_p(\infty) = p$. Assume that k is an even positive integer. Then every $f \in S_k(\Gamma_0(N), \chi)$ has the Fourier expansion $(f|[\alpha_p]_k)(z) = \sum_{n=1}^{\infty} a_n^{(p)} e^{2\pi i n z/\beta}$ at p for some $\beta > 0$. The numbers $\{a_n^{(p)}\}_{n=1}^{\infty}$ are called the Fourier coefficients of f at p.

When we apply Rankin's method to the Dirichlet series corresponding to an automorphic form in $S_k(\Gamma_0(N), \chi)$, certain explicit relations between the Fourier coefficients at all cusps are needed. In this paper, we deal with the problem: Given the coefficients at one cusp, can all coefficients at other cusps be determined? Recently, by using the W matrix in Atkin-Lehner [2] and Hecke operators, Asai [1] solved the problem positively in the case where N is square-free. If N is not square-free, we cannot immediately apply his argument.

In §1, by a different method, we give an affirmative answer to the above problem in the case $N = 4q$ with q prime. §2 and §3 are preparatory sections, where we describe certain properties of Eisenstein-Epstein functions and Maass' theta functions and, in the last section, we give an application of the result in §1 to the Doi-Naganuma lifting in the case of $\boldsymbol{Q}(\sqrt{4q})$ with a prime $q \equiv 3 \pmod 4$. The basic references for this subject are Asai [1], Doi-Naganuma [3], Naganuma [5], Shimura [7] and Zagier [8].

1. Fourier coefficients at various cusps. Throughout this paper, we use the following notations. Let N be a positive integer and let χ be a Dirichlet character modulo N. Put

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, c \equiv 0 \bmod N \right\} .$$

We let $\mathfrak{H}$ denote the complex upper half plane. Assume that f is a holomorphic function on $\mathfrak{H}$. Put $(f|[\sigma]_k)(z) = (\det \sigma)^{k/2}(cz + d)^{-k} f(\sigma(z))$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\boldsymbol{R})$. We denote by $S_k(\Gamma_0(N), \chi)$ and by $S_k^0(\Gamma_0(N), \chi)$