Tôhoku Math. Journ.
32 (1980), 505-510.

# NOTES ON THE STABILITY IN VARIATION

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

JUNJI KATO AND SADAO KUROSAKI

(Received June 21, 1979)

For linear differential equations it is obvious that the stability of an arbitrary solution is equivalent to the stability of the zero solution. However, the situation will be quite different for nonlinear equations. Such a problem on the stability of each of solutions was posed and discussed by several authors as extreme stability of the system [5], [10], (positively) equi-continuous flow [8], stability in variation [2], [3], [6], [9], and so on.

To study perturbation problems of nonlinear differential equations, Alekseev's formula of constant variations [1] presents a useful approach, and it is still more powerful if one assumes some sorts of stability in variation. (cf. [2], [3], [6], [7], [9].) On the other hand, it is also well-known that Liapunov's second method is an effective tool for perturbation problems (see [10]). Leela [6] and Kamala-Lakshmikantham [3] gave a way to construct desirable Liapunov functions under the stability in variation. However, there was an inaccuracy in their proof, as we see in Remark 3.

The purpose of this note is to give characterizations of the stability in variation in the two ways, one by constructing a Liapunov function and the other by generalizing the idea in [4].

Let $R^n$ be the Euclidean $n$-space with a norm $\|\cdot\|$, and set $S_\alpha = \{x \in R^n : \|x\| \leqq \alpha\}$ for a given $\alpha \geqq 0$. For any matrix $A$, $\|A\|$ denotes the associated norm defined by $\sup_{\|x\| \leqq 1}\|Ax\|$.

Consider the equations

$$\dot{x} = f(t, x)\ , \tag{1}$$

where $f(t, x)$ is continuous on $I \times R^n$, $I = [0, \infty)$, and continuously differentiable with respect to $x$. Let $x(t, s, \xi)$ be the (unique) solution of (1) through $(s, \xi)$, and put $\Phi(t, s, \xi) = (\partial/\partial\xi)x(t, s, \xi)$, which is well-defined under the assumptions. It is clear that $\Phi(t, s, \xi)$ is the fundamental matrix of the variational equations