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BEHAVIOR OF SOLUTIONS FOR THE LINEARIZED NAVIER-STOKES EQUATIONS WITH VANISHING VISCOSITY

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1. Introduction. Let Ω be a bounded domain in E_3 with smooth boundary $\partial\Omega$ and let $T>0$ be fixed. For a vector function $\boldsymbol{v}_\nu=(v_{\nu,1}, v_{\nu,2}, v_{\nu,3})$ and a scalar function p_ν representing the velocity of the fluid and the pressure we consider in $\Omega\times[0, T]$ an initial-boundary value problem for the linearized Navier-Stokes equation:

$$(1.1)\qquad \begin{cases} D_t\boldsymbol{v}_\nu-\nu\Delta\boldsymbol{v}_\nu+\nabla p_\nu=\boldsymbol{f}\,, \\ \operatorname{div}\boldsymbol{v}_\nu=0\,, \\ \boldsymbol{v}_\nu|_{t=0}=\boldsymbol{a}\,,\quad \boldsymbol{v}|_{\partial\Omega}=\boldsymbol{0}\,, \end{cases}$$

where $\boldsymbol{a}(x)$, $\boldsymbol{f}(x, t)$ are given vector functions and ν is the so-called viscosity coefficient.

The existence and uniqueness results for (1.1) are now well known (see, for instance, [1]); whereas the behavior of the solutions, as the viscosity coefficient ν tends to zero, is not yet fully understood, and we wish to study such a question in this note.

We first introduce our notation. Let $L^2(\Omega)$ be the Hilbert space of square integrable real functions on Ω and let $W^{k,2}(\Omega)$ be the Sobolev space of functions with square integrable derivatives up to the order k in $L^2(\Omega)$. For vector valued functions $\boldsymbol{v}=(v_1, v_2, v_3)$ the corresponding spaces are denoted by $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{W}^{k,2}(\Omega)$. The norms will be denoted by $\|\ \|_{L^2(\Omega)}$, $\|\ \|_{W^{k,2}(\Omega)}$ etc. Let $\boldsymbol{C}_0^\infty(\Omega)=\{\boldsymbol{v}=(v_1, v_2, v_3);\ v_i\in C^\infty(\Omega),\ \operatorname{supp}(v_i)\subset\Omega,\ i=1, 2, 3\}$ and $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)=\{\boldsymbol{v}\in\boldsymbol{C}_0^\infty(\Omega); \operatorname{div}\boldsymbol{v}=0\}$. We define $\boldsymbol{L}_\sigma^2(\Omega)$ and $\boldsymbol{H}(\Omega)$ as the closures of $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{W}^{1,2}(\Omega)$. The orthogonal projection from $\boldsymbol{L}^2(\Omega)$ (resp. $\boldsymbol{L}^2(\Omega\times(0, T))=L^2(0, T; \boldsymbol{L}^2(\Omega)))$ onto $\boldsymbol{L}_\sigma^2(\Omega)$ (resp. $L^2(0, T; \boldsymbol{L}_\sigma^2(\Omega)))$ will be denoted by $\boldsymbol{P}_\sigma$.

We now assume in (1.1) that $\boldsymbol{a}\in\boldsymbol{H}(\Omega)$ and $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega\times(0, T))$. Then as will be seen in the next section, we have the solution $(\boldsymbol{v}_\nu, p_\nu)$, which as $\nu\to 0$ converges weakly in the Hilbert space $\boldsymbol{L}^2(\Omega\times(0, T))\times L^2(\Omega\times(0, T))$ to the solution $(\boldsymbol{v}_0, p_0)$, given by