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A REMARK ON THE HYPERBOLIC COLLAR LEMMA

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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1. Preliminaries. Let G be a discrete subgroup of $PSL(2, \boldsymbol{C})$ acting on $H^3 = \{z + uj; z \in \boldsymbol{C}, u > 0\}$, the upper half space model of the hyperbolic space. If $X \in G - \{\text{id.}\}$ is not a parabolic element, then we denote by g_X the geodesic in H^3 joining the fixed points of X on the boundary of H^3. For a positive number k, we define a tubular neighborhood about g_X as the set

$$N_k(X) = \{x \in H^3;\ d(x, g_X) \leqq k\}\ ,$$

where d is the hyperbolic distance. Let G_X be the subgroup of G which leaves g_X invariant. We call $N_k(X)$ a collar for X in G, if $T(N_k(X)) \cap N_k(X) = \emptyset$ for all $T \in G - G_X$ and $T(N_k(X)) = N_k(X)$ for all $T \in G_X$. In this case, the number k is called the width of the collar $N_k(X)$.

The first purpose of this note is to prove the following theorem, the so-called collar lemma.

THEOREM. *Let $G \subset PSL(2, \boldsymbol{C})$ be a non-elementary discrete group.*

(i) *Suppose that $X \in G$ satisfies $0 < |\mathrm{trace}^2 X - 4| = s < s_0 = 2(-1 + \sqrt{2})$. Then g_X has a collar $N_{k(s)}(X)$, where*

$$\sinh^2 k(s) = s^{-1}(1 - s)^{1/2} - 1/2\ . \tag{1}$$

(ii) *Let X and Y be in G and suppose that X and Y generate a non-elementary group. If $0 < |\mathrm{trace}^2 X - 4|$ and $|\mathrm{trace}^2 Y - 4| < 2(-1 + \sqrt{2})$, then the collars $N_{k(s)}(X)$ for X and $N_{k(s')}(Y)$ for Y are disjoint, where $s = |\mathrm{trace}^2 X - 4|$, $s' = |\mathrm{trace}^2 Y - 4|$ and k is the function defined by* (1).

Brooks and Matelski [2] proved the above theorem for the constant $s_0 = 1/2$ and for the function k defined by $\sinh^2 k(s) = s^{-1} - 3/2$. Gallo [3] also obtained the theorem for the constant $s_0 = (\sqrt{41} - 5)/2$ and for k defined by $\sinh^2 k(s) = s^{-1} - (s + 3)/2$. The constant s_0 and the function k in the Theorem are better than those in [2] and [3].

Sections 2 through 4 are devoted to preliminary discussions for the

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