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THE JACOBIANS AND THE DISCRIMINANTS OF FINITE REFLECTION GROUPS

HIROAKI TERAO

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1. Introduction. Let $\boldsymbol{K}$ be a field with characteristic not equal to two. Let V be an l-dimensional vector space over $\boldsymbol{K}$. An invertible linear transformation g on V is called a *reflection* if $\ker(1-g)$ is of codimension one. In this note we study a finite subgroup $G \subseteq GL(V)$ generated by reflections, which is called a *finite reflection group*. Throughout this paper assume that *the order of G is not divisible by the characteristic of $\boldsymbol{K}$*. The aim of this paper is to do an algebraic study of the Jacobian J and the discriminant δ of a finite reflection group, especially of their relations with the derivations. When $\boldsymbol{K}$ has characteristic zero, one of the most powerful techniques to study finite reflection groups is the Molien series. Since the Molien series is not effective for positive characteristics, we have to find another way to get results. Sometimes we can simplify the proofs for characteristic zero by avoiding the Molien series as we will see in this paper.

Let $S=S(V^*)$ be the symmetric algebra of the dual space V^* of V. Then S can be regarded as the ring of polynomial functions on V. Identify S with $\boldsymbol{K}[x_1, \cdots, x_l]$ using a basis $\{x_1, \cdots, x_l\}$ for V^*. Agree that $\deg(x)=1$ for all $x \in V^*-\{0\}$. Since the reflection group G acts on V^* contragrediently, it also acts on $S=S(V^*)$. Let $R=S^G$ be the invariant subring of S under the action of G. By Chevalley's theorem [4], [3: Ch. 5, Sect. 5.5, Th. 4], we know that the invariant subring R is a polynomial graded $\boldsymbol{K}$-algebra, in other words, there exist algebraically independent homogeneous polynomials $f_1, \cdots, f_l \in S$ such that $R=\boldsymbol{K}[f_1, \cdots, f_l]$. The polynomials $f_1, \cdots, f_l$ are called *basic invariants* of G. Although the choice of basic invariants is not unique, their Jacobian $J=\det[\partial f_i/\partial x_j]_{1 \leqq i, j \leqq n}$ is unique up to a constant multiple. Let $\delta \in R$ be a generator of the ideal $JS \cap R$ (JS stands for the principal ideal of S generated by J). This δ is called the *discriminant* of G. The discriminant $\delta \in R$ is also unique up to a constant multiple.

In general, let A be an arbitrary $\boldsymbol{K}$-algebra. Let $\operatorname{Der}(A)$ be the module of $\boldsymbol{K}$-derivations of A:

$$\operatorname{Der}(A)=\{\theta\colon A \to A \mid \theta \text{ is } \boldsymbol{K}\text{-linear and } \theta(fg)=f\theta(g)+g\theta(f) \text{ for all } f, g \in S\}\,.$$

For $f \in A$, an A-submodule $D_A(f)$ of $\operatorname{Der}(A)$ is defined by

$$D_A(f)=\{\theta \in \operatorname{Der}(A) \mid \theta(f) \in fA\}\,.$$

This is an algebraic version of logarithmic vector fields along $\{f=0\}$ (cf. [8], [9]). The finite reflection group G naturally acts on $\operatorname{Der}(S)$. Let $\operatorname{Der}(S)^G$ be the R-module of G-