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PARTIAL THETA FUNCTION EXPANSIONS

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Introduction. In a series of recent papers, Andrews [1], [2] discussed in detail certain groups of formulae, which he found in Ramanujan's "Lost" Notebook (as he preferred to call it) and has given remarkable and ingeneous proofs for some of Ramanujan's tricky and mysterious indentities.

In the second paper of the series he considered the idea of expanding θ-functions in terms of partial theta functions. He considered the four families of trigonometric polynomials:

$$\theta_{1;N}(Z;q)=2q^{1/4}\operatorname{Sin} Z\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1-2q^{2m}\operatorname{Cos}2Z+q^{4m}), \tag{1.1}$$

$$\theta_{2;N}(Z;q)=2q^{1/4}\operatorname{Cos} Z\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1+2q^{2m}\operatorname{Cos}2Z+q^{4m}), \tag{1.2}$$

(introduced by Watson [5, p. 67])

$$\theta_{3;N}(Z;q)=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1+2q^{2m-1}\operatorname{Cos}2Z+q^{4m-2}), \tag{1.3}$$

$$\theta_{4;N}(Z;q)=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1-2q^{2m-1}\operatorname{Cos}2Z+q^{4m-2}). \tag{1.4}$$

These trigonometric polynomials are partial products of the four classical theta functions, first treated extensively by Jacobi. These are

$$\theta_1(Z;q)=2\prod_{n=1}^{\infty}(1-q^{2n})q^{1/4}\operatorname{Sin} Z\prod_{m=1}^{\infty}(1-2q^{2m}\operatorname{Cos}2Z+q^{4m}), \tag{1.5}$$

$$\theta_2(Z;q)=2\prod_{n=1}^{\infty}(1-q^{2n})q^{1/4}\operatorname{Cos} Z\prod_{m=1}^{\infty}(1+2q^{2m}\operatorname{Cos}2Z+q^{4m}), \tag{1.6}$$

$$\theta_3(Z;q)=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{\infty}(1+2q^{2m-1}\operatorname{Cos}2Z+q^{4m-2}), \tag{1.7}$$

$$\theta_4(Z;q)=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{\infty}(1-2q^{2m-1}\operatorname{Cos}2Z+q^{4m-2}). \tag{1.8}$$

These can be alternatively expressed as