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AFFINE HYPERSURFACES WITH PARALLEL CUBIC FORMS

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In this note, we first investigate the symmetry properties of the tensors ∇C and $\nabla^2 C$, where ∇ is the induced affine connection and C is the cubic form of a nondegenerate affine hypersurface M^n in $\boldsymbol{R}^{n+1}$. In particular, we study hypersurfaces with parallel cubic form, i.e. $\nabla C=0$. In the case $n=2$, this condition is known to characterize a Cayley surface (Nomizu and Pinkall [3]). We obtain a certain class of more general affine surfaces and hypersurfaces.

On the other hand, for an affine hypersurface M^n, $n\geq 3$, condition $\nabla R=0$ (i.e. parallel curvature tensor field) implies that M^n is an improper affine hypersphere or a quadric (Verheyen and Verstraelen [7]). We shall provide a generalization of this result by proving that the condition $\nabla^2 R=0$ implies $\nabla R=0$ for an affine hypersurface. Recall that, for a Riemannian manifold, the condition $\nabla^2 R=0$ (in fact, $\nabla^k R=0$ for some integer k) implies $\nabla R=0$ but that such a result does not hold for an affine connection in general.

Our study shows the common background for these results on the covariant differentials of the cubic form and those of the curvature tensor field.

1. Preliminaries. Although we mostly follow the notation in [3] in this paper, we consider exclusively the classical theory of nondegenerate affine hypersurfaces M^n in $\boldsymbol{R}^{n+1}$ in the sense of Blaschke (see [1], [2], [5] and [6]).

The difference between the induced affine connection ∇ and the Levi-Civita connection $\hat{\nabla}$ for the affine metric h is denoted by K:

(1) $$K_X=\nabla_X-\hat{\nabla}_X$$

and we also write

(2) $$K_X Y=K_Y X=K(X, Y)\,.$$

The so-called apolarity condition can be expressed by

(3) $$\operatorname{trace} K_X=0 \qquad \text{for every} \quad X\,.$$

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