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UNIFORM ULTIMATE BOUNDEDNESS AND PERIODICITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction. In this section we give a brief description of the background of and solution to the problem considered. The detailed assumptions are given in the next section.

It is shown that if solutions of the infinite delay T-periodic system

$$x' = f(t, x_t) \tag{1}$$

are uniformly ultimately bounded (UUB) in the *supremum norm*, then there is a T-periodic solution. This improves known results which have required that solutions of (1) also be uniformly bounded (UB). It was shown by Kato [11] that uniform ultimate boundedness for (1) does not imply uniform boundedness.

This problem goes back to Levinson [12]. It was solved for second order ordinary differential equations by Cartwright [7] and Massera [13]; solutions for general n followed from Browder's fixed point theorem (cf. Browder [2] and Yoshizawa [16; p. 158]). Hale and Lopes [9] show that if (1) has finite delay then UB and UUB imply that (1) has a T-periodic solution. It is known that for periodic ordinary differential equations then UUB implies UB.

When (1) has unbounded delay, an example of Seifert [15] shows that if UUB is expected, then (1) must have some type of fading memory. Moreover, it was believed until very recently that in order to prove that (1) has a T-periodic solution using UB and UUB, then the boundedness must be in terms of a weighted norm on the phase space [1] which allowed unbounded initial functions. If (1) has the special form

$$x' = h(t, x) + \int_{-\infty}^{t} q(t, s, x(s))ds \tag{1*}$$

then a simple fading memory was defined in [3] which enables one to show that if solutions are UB and UUB in the supremum norm, then the same is true for a weighted norm. Investigators have been unsuccessful in extending the stated result for (1)* to (1). The details for this summary are found in [4; pp. 214–324]. The recent survey book by Hale [8] continues the problem to operator equations.

Recently, Burton-Dwiggins-Feng [5] have shown that if (1) has a fading memory

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