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BUBBLING OUT OF EINSTEIN MANIFOLDS

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In [1], [8], and [4] the following compactness theorem of the space of Einstein metrics is obtained in the spirit of Gromov theory.

THEOREM A. *Let (X_i, g_i) be a sequence of n-dimensional $(n\geq 4)$ smooth manifolds and Einstein metrics on them with uniformly bounded Einstein constants $\{e_i\}$ satisfying*

$$\operatorname{diam}(X_i, g_i)\leq D\,, \quad \operatorname{vol}(X_i, g_i)\geq V \quad \textit{and} \quad \int_{X_i} |R_{g_i}|^{n/2} dV_i \leq R$$

for some positive constants D, V and R, where we denote the curvature tensor of a metric g by R_g. Then there exist a subsequence $\{j\}\subset\{i\}$ and a compact Einstein orbifold (X_∞, g_∞) with a finite set of singular points $S=\{x_1, x_2, \cdots, x_s\}\subset X_\infty$ (possibly empty) for which the following statements hold:

(1) *(X_j, g_j) converges to (X_∞, g_∞) in the Hausdorff distance.*

(2) *There exists an into diffeomorphism $F_j\colon X_\infty\setminus S\to X_j$ for each j such that $F_j^* g_j$ converges to g_∞ in the C^∞-topology on $X_\infty\setminus S$.*

(3) *For every $x_a\in S$ $(a=1, 2, \cdots, s)$ and j, there exist $x_{a,j}\in X_j$ and a positive number r_j such that*

(3.a) *$B(x_{a,j}; \delta)$ converges to $B(x_a; \delta)$ in the Hausdorff distance for all $\delta>0$.*

(3.b) $\lim_{j\to\infty} r_j=0$.

(3.c) *$((X_j, r_j^{-2}g_j), x_{a,j})$ converges to $((M_a, h_a), x_{a,\infty})$ in the pointed Hausdorff distance, where (M_a, h_a) is a complete, non-compact, Ricci-flat, non-flat n-manifold which is* ALE, *of order $n-1$ in general, of order n if (M_a, h_a) is Kähler or $n=4$.*

(3.d) *There exists an into diffeomorphism $G_j\colon M_a\to X_j$ such that $G_j^*(r_j^{-2}g_j)$ converges to h_a in the C^∞-topology on M_a.*

(4) *It holds that*

$$\lim_{j\to\infty}\int_{X_j} |R_{g_j}|^{n/2} dV_j \geq \int_{X_\infty} |R_{g_\infty}|^{n/2} dV_\infty + \sum_a \int_{M_a} |R_{h_a}|^{n/2} dV_{h_a}\,.$$

Moreover if (X_i, g_i) are Kähler, then (X_∞, g_∞) and (M_a, h_a) are also Kähler.

Here we call a smooth n-dimensional complete Riemannian orbifold (X, g) asymptotically locally Euclidean (ALE, for short) of order $\tau>0$, if there exists a compact subset $K\subset X$ such that $X\setminus K$ has coordinates at infinity; namely there are $R>0$, $0<\alpha<1$, a finite subgroup $\Gamma\subset O(n)$ acting freely on $\boldsymbol{R}^n\setminus B(0; R)$, and a C^∞-diffeomorphism