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GALOIS EXTENSIONS OF ALGEBRAIC FUNCTION FIELDS

Dedicated to Professor Ichiro Satake on his sixtieth birthday

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Introduction. Let K be an algebraic function field of one variable of genus g over the complex number field $\boldsymbol{C}$. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ and $m_1, \cdots, m_n$ be any given distinct prime divisors of K over $\boldsymbol{C}$ and any given elements of $\boldsymbol{N} \cup \{\infty\}$ satisfying $2g-2+\sum_{i=1}^{n}(1-1/m_i)>0$, respectively, where $\boldsymbol{N}$ means the set of natural numbers. Set $\Delta=\{i \mid 1 \leqq i \leqq n, m_i=\infty\}$ and $\Delta'=\{1, 2, \cdots, n\}-\Delta$. We consider all Galois extensions L of K in a fixed algebraic closure $\bar{K}$ of K such that the divisors $\mathfrak{D}(L/K)'$ divide $\sum_{i=1}^{n}(m_i-1)\mathfrak{P}_i$ and that the ramification indices $e_{\mathfrak{P}_i}$ $(1 \leqq i \leqq n)$ of divisors of L over $\mathfrak{P}_i$ divide m_i, where $\mathfrak{D}(L/K)'=\mathfrak{D}(L/K)-\sum_{i \in \Delta} v_{\mathfrak{P}_i}\mathfrak{P}_i$ and $\mathfrak{D}(L/K)=\sum_{\mathfrak{P}} v_{\mathfrak{P}}\mathfrak{P}$ is the ramification divisor of L over K. It is well known that the Galois group of the composite field of all these Galois extensions of K is isomorphic to the profinite completion of a Fuchsian group Γ_0 with signature $(m_1, \cdots, m_n : g)$ (cf. Eichler [2] and Weil [9]). We fix an odd prime number p and denote by $\boldsymbol{F}_p$ the finite field with p elements.

In this paper, we shall study the number of Galois extensions L (resp. $\tilde{L}$) of K in $\bar{K}$ with $SL_2(\boldsymbol{F}_p)$ (resp. $PSL_2(\boldsymbol{F}_p)$) as their Galois groups such that $\mathfrak{D}(L/K)' \mid \sum_{i=1}^{n}(m_i-1)\mathfrak{P}_i$, $e_{\mathfrak{P}_i} \mid m_i$ $(i \in \Delta')$ and $e_{\mathfrak{P}_1}=m_1$. This number is independent of $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$. So we denote by $N(m_1, \cdots, m_n: g)$ (resp. $\tilde{N}(m_1, \cdots, m_n: g)$) the number of such Galois extensions. Throughout this paper, for technical reasons, we confine ourselves to the case where $m_1=p$. This assumption is essential to our arguments. In Theorem 1, we shall obtain formulas for $N(m_1, \cdots, m_n: g)$ and $\tilde{N}(m_1, \cdots, m_n: g)$. In particular, $N(k)=N(p, \infty, k: 0)$ $(k=1, \cdots, 7$ or k is a prime$)$, $N(p, \overbrace{\infty, \cdots, \infty}^{n-1} : g)$ and $\tilde{N}(p, \overbrace{\infty, \cdots, \infty}^{n-1} : g)$ $(p \neq 2)$ can be determined explicitly as follows:

$$N(1)=N(2)=0,\ N(4)=(p-1),\ N(6)=2(p-1)$$

$$N(q)=\begin{cases} (p-1)(q-1)/2 & q \mid (p^2-1) \\ p-1 & q=p \\ 0 & \text{otherwise} \end{cases}$$

for every odd prime q. Furthermore,

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