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HYPERSURFACE SIMPLE $K3$ SINGULARITIES

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Introduction. In the theory of two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the next most reasonable class of singularities after rational singularities. Cusp singularities appear on the Satake compactifications of Hilbert modular surfaces and have loops of rational curves as the exceptional sets of the minimal resolution. Simple elliptic singularities were investigated by Saito [11] in detail. By definition, each of them has a nonsingular elliptic curve as the exceptional set of the minimal resolution. Here we are interested especially in a hypersurface *simple elliptic* singularity (X, x). In this case, the defining equation of (X, x) is given by one of the following in some coordinates z_1, z_2, z_3 around x,

$$\tilde{E}_6 : z_1^3 + z_2^3 + z_3^3 + \lambda_1 z_1 z_2 z_3 = 0 \qquad (E^2 = -3)\,,$$

$$\tilde{E}_7 : z_1^2 + z_2^4 + z_3^4 + \lambda_2 z_1 z_2 z_3 = 0 \qquad (E^2 = -2)\,,$$

$$\tilde{E}_8 : z_1^2 + z_2^3 + z_3^6 + \lambda_3 z_1 z_2 z_3 = 0 \qquad (E^2 = -1)\,,$$

with the parameter satisfying $\lambda_1^3 + 27 \neq 0$, $\lambda_2^4 - 64 \neq 0$, $\lambda_3^6 - 432 \neq 0$ and corresponding to the moduli of the elliptic curve E which appears as the exceptional set.

The purpose of this paper is to study similar properties for simple $K3$ singularities which we regard as natural generalizations in three-dimensional case of simple elliptic singularities.

The notion of a simple $K3$ singularity was defined by Watanabe [4] as a three-dimensional Gorenstein purely elliptic singularity of (0, 2)-type, whereas a simple elliptic singularity is a two-dimensional purely elliptic singularity of (0, 1)-type. Ishii [4] pointed out that a simple $K3$ singularity is characterized as a quasi-Gorenstein singularity such that the exceptional set of any minimal resolution is a normal $K3$ surface. Let $f \in \boldsymbol{C}[z_0, z_1, z_2, z_3]$ be a polynomial which is nondegenerate with respect to its Newton boundary $\Gamma(f)$ in the sense of [14], and whose zero locus $X = \{f = 0\}$ in $\boldsymbol{C}^4$ has an isolated singularity at the origin $0 \in \boldsymbol{C}^4$. Then the condition for $(X, 0)$ to be a simple $K3$ singularity is given by a property of the Newton boundary $\Gamma(f)$ of f (cf. Proposition 1.6). Tomari [12] showed that a minimal resolution $\pi : (\tilde{X}, E) \to (X, 0)$ of a simple $K3$ singularity is also obtained from $\Gamma(f)$. In this paper, we classify nondegenerate hypersurface simple $K3$ singularities and study the singularities on the $K3$ surface E through the minimal resolution π.

In §2, we classify nondegenerate hypersurface simple $K3$ singularities into ninety