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GLOBAL DENSITY THEOREM FOR A FEDERER MEASURE

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Abstract. The local and global density theorems for the Lebesgue measure in a Euclidean space play a fundamental role in calculus. On the other hand Federer [5] proved a local density theorem for a measure with a doubling condition on a metric space.

The aim of this paper is to prove a global density theorem for a measure with a doubling condition and a class of integrable functions on a metric space. As a special case this theorem also gives a simple and constructive proof to Federer's local density theorem.

A typical example of the above measures is the Hausdorff measure on a self-similar set.

1. Introduction. Throughout the paper $E=(E, d)$ denotes a metric space, $B(x, r)$ for $x\in E$ and $r>0$ the closed ball $\{y\in E; d(x, y)\leq r\}$ and $U(x, r)$ the open ball $\{y\in E; d(x, y)<r\}$, and λ a measure defined on a σ-algebra $\mathscr{B}_\lambda$ of subsets of E such that $\mathscr{B}_\lambda$ includes the Borel field $\mathscr{B}(E)$ of E,

$$\lambda(A)=\inf\{\lambda(G); A\subset G, G \text{ open}\}, \qquad A\in\mathscr{B}_\lambda,$$

and $\lambda(B(x, r))<\infty$ for any $r>0$ and λ-almost all $x\in E$.

For a real measure μ on $(E, \mathscr{B}(E))$, $(d\mu/d\lambda)(x)$ denotes the Radon-Nikodym derivative *in the sense of the Lebesgue decomposition of* μ *with respect to* λ, that is,

$$d\mu(x)=\frac{d\mu}{d\lambda}(x)d\lambda(x)+d\mu_s(x),$$

where $d\mu_s(x)$ is singular with respect to $d\lambda(x)$.

When λ is the Lebesgue measure, the following density theorems are well-known and fundamental in calculus.

THEOREM 1 (Local density theorem, see for example Dunford and Schwartz [4]). *Let λ be the Lebesgue measure on $E=\boldsymbol{R}^n$. Then we have*

$$\frac{d\mu}{d\lambda}(x)=\lim_{r\downarrow 0}\frac{\mu(B(x, r))}{\lambda(B(x, r))}, \quad a.e.\ (d\lambda),$$

for any real measure μ on $\boldsymbol{R}^n$.

THEOREM 2 (Global density theorem). *Let λ be the Lebesgue measure on $E=\boldsymbol{R}^n$,*

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