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GROUPS GRADED BY FINITE ROOT SYSTEMS[1]

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Abstract. A Steinberg group $\mathrm{St}(\Delta, R)$ is defined by the data of a ring R and a root system Δ. This paper aims to study the relationship between the group-theoretic structure of a Steinberg group and the associated ring. We introduce graded groups which are groups satisfying some axioms that are basic properties of $\mathrm{St}(\Delta, R)$, and then show that these properties suffice to determine the structures of graded groups, *by constructing a ring out of a graded group*. Also the central extensions of graded groups are studied.

Introduction. In this paper, the groups graded by finite root systems Δ, or Δ-graded groups, are introduced. These are analogues of Lie algebras graded by finite root systems which are studied by Berman and Moody [1]. The background is the structures of Steinberg groups and Chevalley groups. The connection among Δ-graded groups, Steinberg groups and central extensions can be seen throughout the article.

Assume that our rings are always associative and with the identity element denoted by 1. For each $l \geq 1$, all $(l+1) \times (l+1)$ invertible matrices over R form the general linear group $GL_{l+1}(R)$. Let E_{ij} be the (i, j) matrix unit of $GL_{l+1}(R)$. Then the elementary group $E_{l+1}(R)$, the subgroup of $GL_{l+1}(R)$ generated by $I + rE_{ij}$ for $r \in R$ and $i \neq j$, models the definition of the Steinberg group $\mathrm{St}(A_l, R)$, where A_l is a type of root systems. Both $\mathrm{St}(A_l, R)$ and E_{l+1} can be assigned a grading by the root system of Type A_l in terms of the group commutators. Now the question is: without given a ring in advance, would the graded property will determine the structure of such a group? This motivates our definition for a Δ-graded group (cf. Definition (2.1)), where we assume that the root system Δ is always one of the types A_l, $l \geq 3$, D_l, $l \geq 4$ and E_l, $l = 6, 7, 8$, unless otherwise stated. We have:

(2.3) THEOREM. *Let G be a group graded by Δ. Then there is an associative ring R with* 1, *such that G is a homomorphic image of the Steinberg group* $\mathrm{St}(\Delta, R)$. *Moreover, R is commutative if Δ is of Type D_l or E_l.*

Note that here all associative rings fit in here. For the proof, the critical point is to define the ring R out of such a group. The main theme of the proof is set in [1] on the Lie algebra level.

Then for each Δ-graded group, we may attach a ring R. A Δ-homomorphism of Δ-graded groups is naturally understood to be a group homomorphism which preserves

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