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HARMONIC MAPS OF NONORIENTABLE SURFACES TO FOUR-DIMENSIONAL MANIFOLDS

TORU ISHIHARA*

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Abstract. We construct explicit harmonic maps of the projective plane or a quotient space of a hyperelliptic Riemann surface into the unit 4-sphere.

1. Introduction. Harmonic maps of nonorientable surfaces are not studied so much (see, for example, [EeL1], [EeL3]). The existence problem of harmonic representatives in homotopy classes of maps of nonorientable surfaces was studied in [Ee12]. Equivariant minimal immersions of the projective plane into S^n or P^n are determined by Ejiri [Eg]. In the present paper, we will try to construct harmonic maps from nonorientable surfaces into 4-dimensional Riemannian manifolds. We deal with a nonorientable surface $\mathscr{M}$ which is a quotient of a Riemann surface M by the equivalent relation $z \sim w$ if and only if $w = I(z)$, where I is an anti-holomorphic involution of M without fixed points. Especially, we will be concerned with the following nonorientable surfaces. We first identify the unit 2-sphere S^2 with $\boldsymbol{C} \cup \{\infty\}$ and put $M = \boldsymbol{C} \cup \{\infty\}$. The map corresponding to the antipodal map is an involution of M given by $I(z) = -1/\bar{z}$. The quotient space is the projective plane. Next, let T_{l-1} be a hyperelliptic Riemann surface given by

$$T_{l-1} = \{(z, w) \in (\boldsymbol{C} \cup \{\infty\})^2;\ w^2 = \prod_{j=1}^{l} (d_j - z)(\bar{d}_j + z)\}, \tag{1.1}$$

where $d_i \neq d_j$ for any $i \neq j$ and $d_i \neq -\bar{d}_j$ for any $i \neq j$. Let $I(z,w) := (-\bar{z}, -\bar{w})$ for $(z, w) \in T_{l-1}$. Then it is an antiholomorphic involution without fixed points (see [11]). Let $P_l := T_{l-1}/\{I\}$ be the quotient space of T_{l-1} by the equivalence relation given by I. Then P_l is a nonorientable surface of genus l. We may regard P_1 as the projective plane and P_2 as the Klein bottle. Now we return to the general setting. Let M be a Riemann surface with involution I and $\pi: M \to \mathscr{M}$ the natural projection of M to the quotient space. A map h of M into a Riemannian manifold N is factored as $h = \mathscr{h} \cdot \pi$, where $\mathscr{h}$ is a map of $\mathscr{M}$ into N, if and only if $h(I(p)) = h(p)$ for each $p \in M$. Let g be a Riemannian metric compatible with the conformal structure of M. We give a natural Riemannian structure $\mathscr{g}$ on $\mathscr{M}$ such that π is locally isometric. Evidently the assign-

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