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CONJUGATE EXPANSIONS FOR ULTRASPHERICAL FUNCTIONS

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Abstract. We define and investigate the Hilbert transform for expansions with respect to the system of ultraspherical functions. Using deep estimates done by Muckenhoupt and Stein in the polynomial expansion case we prove the existence of boundary values of the conjugate Poisson integrals of integrable functions. The limit function then satisfies usual L^p and weak type (1, 1) estimates.

1. Introduction. Thirty years ago in their classical paper [6] Muckenhoupt and Stein investigated the ultraspherical expansions from the harmonic analysis point of view. One of the main results they proved was that the conjugacy mapping $f \mapsto \tilde{f}$ is a bounded operator on L^p, $1<p<\infty$. Here the definition of conjugacy is insightfully introduced in such a way that the Poisson integral and the conjugate Poisson integral corresponding to f are related by suitable Cauchy-Riemann equations. More specifically, if $f \in L^1((0, \pi), (\sin\theta)^{2\lambda}d\theta)$, $\lambda>0$, has the expansion $\sum a_n P_n^\lambda(\cos\theta)$ then its conjugate $\tilde{f}$ is formally defined by

$$\sum \frac{2\lambda}{n+2\lambda} a_n \sin\theta \cdot P_{n-1}^{\lambda+1}(\cos\theta)\,.$$

Then the Poisson integral

$$f(r, \theta)=\sum a_n r^n P_n^\lambda(\cos\theta)$$

and the conjugate Poisson integral

$$\tilde{f}(r, \theta)=\sum \frac{2\lambda}{n+2\lambda} a_n r^n \sin\theta \cdot P_{n-1}^{\lambda+1}(\cos\theta)$$

satisfy

$$\frac{\partial}{\partial r}((r\sin\theta)^{2\lambda}\tilde{f})=-r^{2\lambda-1}(\sin\theta)^{2\lambda}\frac{\partial f}{\partial\theta}\,,$$

$$\frac{\partial}{\partial\theta}((r\sin\theta)^{2\lambda}\tilde{f})=r^{2\lambda+1}(\sin\theta)^{2\lambda}\frac{\partial f}{\partial r}\,.$$

In [4], [5] Muckenhoupt proved L^p conjugate function theorems for Hermite and

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