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INFINITESIMAL ISOMETRIES OF FRAME BUNDLES WITH A NATURAL RIEMANNIAN METRIC II

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Abstract. We consider the bundle of all oriented orthonormal frames over an orientable Riemannian manifold. This bundle has a natural Riemannian metric which is defined by the Riemannian connection of the base manifold. The purpose of the present paper is to clarify the structure of the Lie algebra of the group of all isometries of the bundle with the Riemannian metric.

1. Introduction. Let $(M, \langle\ ,\ \rangle)$ be a connected orientable Riemannian manifold of dimension $n \geqq 2$ and $SO(M)$ the bundle of all oriented orthonormal frames over M. $SO(M)$ has a Riemannian metric $\langle\ ,\ \rangle$ defined naturally as follows:

$$\begin{aligned}\langle X, Y\rangle &= \langle \theta(X), \theta(Y)\rangle + \langle \omega(X), \omega(Y)\rangle \\ &= {}^t(\theta(X))\theta(Y) + \operatorname{trace}({}^t(\omega(X))\omega(Y)),\end{aligned}$$

where ω and θ are the Riemannian connection form and the canonical form on $SO(M)$, respectively.

In [5], we gave a decomposition of a Killing vector field on $(SO(M), \langle\ ,\ \rangle)$ which is fiber preserving (see Proposition A of § 2) and we proved that M has constant curvature $1/2$, if $(SO(M), \langle\ ,\ \rangle)$ admits a horizontal Killing vector field which is not fiber preserving (see Proposition B of § 2). In the present paper, we give a decomposition of an arbitrary Killing vector field on $(SO(M), \langle\ ,\ \rangle)$ under the assumption that M is complete. The result is stated in the following theorem.

Let p be the projection $SO(M) \to M$. The canonical form θ is an $\boldsymbol{R}^n$-valued 1-form defined by $\theta_u(X) = u^{-1} \circ p(X)$, where u is regarded as a linear isometry of $(\boldsymbol{R}^n, \langle\ ,\ \rangle)$ onto the tangent space at $p(u)$. Let $\mathfrak{o}(n)$ be the Lie algebra of the special orthogonal group $SO(n)$. For each $A \in \mathfrak{o}(n)$, we define a vector field A^* on $SO(M)$ by $\omega(A^*) = A$ and $\theta(A^*) = 0$. A^* is called the fundamental vector field corresponding to A. For each $\xi \in \boldsymbol{R}^n$, we define a vector field $B(\xi)$ on $SO(M)$ by $\omega(B(\xi)) = 0$ and $\theta(B(\xi)) = \xi$. $B(\xi)$ is called the standard horizontal vector field corresponding to ξ. Let ϕ be a 2-form on M and F the tensor field of type (1, 1) on M defined by $\langle FY, Z\rangle = \phi(Y, Z)$. We define an $\mathfrak{o}(n)$-valued function $F^\sharp$ on $SO(M)$ and a vector field ϕ^L or F^L on $SO(M)$ by $F^\sharp(u) =$

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