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0-CYCLES ON THE ELLIPTIC MODULAR SURFACE OF LEVEL 4

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Abstract. We prove a finiteness result on the torsion subgroup in the Chow group of zero cycles on the elliptic modular surface of level four. The main ingredient is Shioda's interpretation of this surface as the Kummer surface associated to the self-product of a certain elliptic curve. On the way we extend the main finiteness theorem on torsion zero cycles on the self-product of a modular elliptic curve to the case where the elliptic curve has complex multiplication and its conductor is a power of a prime.

Introduction. In this paper we will provide a new example for the finiteness of torsion zero cycles on an algebraic surface defined over a number field. Let B' be the subvariety in $\boldsymbol{P}^2 \times (\boldsymbol{P}^1 \setminus \Sigma)$, $\Sigma = (0, \infty, \pm 1, \pm i)$, defined by the equation

$$y^2 = x(x-1)\left(x - \frac{1}{2}\left(\sigma + \frac{1}{\sigma}\right)^2\right)$$

where (x, y) and σ are the inhomogeneous coordinates of $\boldsymbol{P}^2$ and of $\boldsymbol{P}^1$. Then B' is a smooth algebraic surface defined over $\boldsymbol{Q}$. Let B be its minimal model. Then B is an elliptic surface over $\boldsymbol{P}^1$. It is evident from the work of Shioda [Shi, Theorem 1] that after base change to the field $K = \boldsymbol{Q}(i)$, B becomes isomorphic to the elliptic modular surface C, which is defined as a suitable compactification of the universal elliptic curve over the modular curve $X(4)$ defined over K, i.e. we have $B \otimes_{\boldsymbol{Q}} K \cong C$. The main results in the paper are the following:

For a variety X let $\mathrm{CH}_0(X)$ be the Chow group of zero cycles modulo rational equivalence and $\mathrm{CH}_0(X)\{p\}$ its p-primary torsion subgroup for a prime p.

THEOREM A. *Let B be as above and p a prime such that $p > 3$. Then $\mathrm{CH}_0(B)\{p\}$ is a finite group.*

For the elliptic modular surface C we have a weaker result which shows that we have at least enough elements in the K-Theory of C to kill cycles in the closed fibers at good reduction primes. Let $\mathscr{C}$ be a proper smooth model of C over $O_K[1/2]$ and $C_\wp$ the closed fiber of $\mathscr{C}$ at the prime $\wp$, $\wp \nmid 2$. Let $\mathrm{CH}^2(\mathscr{C})$ be the Chow group of codimension 2 of $\mathscr{C}$. Then we have:

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