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ON A TWISTED DE RHAM COMPLEX

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Abstract. We show that, given a projective regular function $f: X \to \boldsymbol{C}$ on a smooth quasi-projective variety, the corresponding cohomology groups of the twisted de Rham complex $(\Omega_X^{\cdot}, d-df\wedge)$ and of the complex $(\Omega_X^{\cdot}, df\wedge)$ have the same dimension. We generalize the result to de Rham complexes with coefficients in a mixed Hodge Module.

Introduction.

0.1. Let $\tilde{X}$ be a smooth projective variety over $\boldsymbol{C}$ and $(\Omega_{\tilde{X}}^{\cdot}, d)$ be the complex of algebraic differential forms. Hodge theory and GAGA theorem of Serre (see also [7] for an algebraic argument, or [8] for other references) show that the hypercohomology spaces on $\tilde{X}$ of both complexes $(\Omega_{\tilde{X}}^{\cdot}, d)$ and $(\Omega_{\tilde{X}}^{\cdot}, 0)$ have the same dimension (this follows from the degeneracy at E_1 of the spectral sequence Hodge $\Rightarrow$ de Rham).

0.2. Denote by $\mathcal{O}_{\tilde{X}}$ the sheaf of regular functions on $\tilde{X}$ and by $\mathcal{D}_{\tilde{X}}$ the sheaf of differential operators with coefficients in $\mathcal{O}_{\tilde{X}}$. More generally, let $(\tilde{M}, F)$ be a mixed Hodge Module on $\tilde{X}$ as defined by M. Saito [18, §4], where $F_{\bullet}\tilde{M}$ is in particular a good filtration (increasing and exhaustive) of the $\mathcal{D}_{\tilde{X}}$-Module $\tilde{M}$. The (algebraic) de Rham complex $\mathrm{DR}(\tilde{M}) = (\Omega_{\tilde{X}}^{\cdot} \otimes_{\mathcal{O}_{\tilde{X}}} \tilde{M}, \nabla)$ is naturally filtered using F (see [3]): the degree-l term of $F_k\mathrm{DR}(\tilde{M})$ is $\Omega_{\tilde{X}}^{l} \otimes F_{k+l}\tilde{M}$, that is also denoted by

$$F_k\mathrm{DR}(\tilde{M}) = \left(\Omega_{\tilde{X}}^{\cdot} \underset{\mathcal{O}_{\tilde{X}}}{\otimes} F_k[-\cdot]\tilde{M}, \nabla\right).$$

The associated graded complex is equal to the complex $(\Omega_{\tilde{X}}^{\cdot} \otimes_{\mathcal{O}_{\tilde{X}}} \mathrm{gr}^F\tilde{M}, \mathrm{gr}^F\nabla)$, where $\mathrm{gr}^F\nabla = \bigoplus_k [\nabla]_k^{k+1}$ and $[\nabla]_k^{k+1}$ is the degree-1 $\mathcal{O}_{\tilde{X}}$-linear morphism induced by ∇

$$\Omega_{\tilde{X}}^{l} \underset{\mathcal{O}_{\tilde{X}}}{\otimes} \mathrm{gr}_k^F\tilde{M} \longrightarrow \Omega_{\tilde{X}}^{l+1} \underset{\mathcal{O}_{\tilde{X}}}{\otimes} \mathrm{gr}_{k+1}^F\tilde{M}.$$

The degeneracy at E_1 (see [18, (4.1.3)]) now implies that the hypercohomology spaces on $\tilde{X}$ of the complexes $\mathrm{DR}(\tilde{M})$ and $(\Omega_{\tilde{X}}^{\cdot} \otimes_{\mathcal{O}_{\tilde{X}}} \mathrm{gr}^F\tilde{M}, \mathrm{gr}^F\nabla)$ have the same dimension.

0.3. Let $\tilde{f}: \tilde{X} \to \boldsymbol{P}^1$ be a morphism of algebraic varieties and let $f: X \to \boldsymbol{A}^1$ be its restriction over the affine line $\boldsymbol{A}^1$. Thus, X is quasi-projective and f is projective.

THEOREM 1. *Let (M, F) be a mixed Hodge Module on X. Then the hypercohomology spaces on X of the complexes $(\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M, \nabla - df\wedge)$ and $(\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} \mathrm{gr}^F M, \mathrm{gr}^F\nabla - df\wedge)$ have the same (finite) dimension.*

0.4. REMARK. If ϕ_f denotes the vanishing cycle functor as defined by Deligne [6] (see also [10]) and $\mathrm{DR}^{\mathrm{an}}$ the analytic de Rham functor, it is well-known (cf. §1.1)

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