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QUASI-EINSTEIN TOTALLY REAL SUBMANIFOLDS OF THE NEARLY KÄHLER 6-SPHERE

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Abstract. We investigate Lagrangian submanifolds of the nearly Kähler 6-sphere. In particular we investigate Lagrangian quasi-Einstein submanifolds of the 6-sphere. We relate this class of submanifolds to certain tubes around almost complex curves in the 6-sphere.

1. Introduction. In this paper, we investigate 3-dimensional totally real submanifolds M^3 of the nearly Kähler 6-sphere S^6. A submanifold M^3 of S^6 is called totally real if the almost complex structure J on S^6 interchanges the tangent and the normal space. It has been proven by Ejiri ([E1]) that such submanifolds are always minimal and orientable. In the same paper, he also classified those totally real submanifolds with constant sectional curvature. Note that 3-dimensional Einstein manifolds have constant sectional curvature. Here, we will investigate the totally real submanifolds of S^6 for which the Ricci tensor has an eigenvalue with multiplicity at least 2. In general, a manifold M^n whose Ricci tensor has an eigenvalue of multiplicity at least $n-1$ is called quasi-Einstein.

The paper is organized as follows. In Section 2, we recall the basic formulas about the vector cross product on $\boldsymbol{R}^7$ and the almost complex structure on S^6. We also relate the standard Sasakian structure on S^5 with the almost complex structure on S^6. Then, in Section 3, we derive a necessary and sufficient condition for a totally real submanifold of S^6 to be quasi-Einstein. Using this condition, we deduce from [C], see also [CDVV1] and [DV], that totally real submanifolds M with $\delta_M = 2$ are quasi-Einstein. Here, δ_M is the Riemannian invariant defined by

$$\delta_M(p) = \tau(p) - \inf K(p)\,,$$

where $\inf K$ is the function assigning to each $p \in M$ the infimum of $K(\pi)$, $K(\pi)$ denoting the sectional curvature of a 2-plane π of T_pM, where π runs over all 2-planes in T_pM and τ is the scalar curvature of M defined by $\tau = \sum_{i<j} K(e_i \wedge e_j)$. Totally real submanifolds of S^6 with $\delta_M = 2$ have been classified in [DV]. Essentially, these submanifolds are either local lifts of holomorphic curves in $\boldsymbol{C}P^2$ or tubes with radius $\pi/2$ in the direction of NN^2, where N^2 is a non-totally geodesic almost complex curve and NN^2 denotes the vector bundle whose fibres are planes orthogonal to the first osculating space of N^2.

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