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FACTORIZATION OF ENTIRE FUNCTIONS

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1. A meromorphic function $F(z) = f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that $f(z)$ is non-linear and meromorphic and $g(z)$ is non-linear and entire (g may be meromorphic when $f(z)$ is rational). $F(z)$ is said to be prime (pseudo-prime) if every factorization of the above form implies that $g(z)$ is linear (a polynomial) unless $f(z)$ is linear (rational). An entire function $F(z)$ is said to be E-prime if it is prime for entire f and g.

Gross [7] posed an open problem whether there exist prime entire periodic functions. In this paper we shall prove the existence of an entire periodic function which is prime (Theorem 2). Our proof is very hard and needs a new idea. We make use of a regular function in $0 < |w| < \infty$. In [7] it was shown that the E-primeness does not imply the primeness. We shall give here another example showing this fact. Our example needs a slightly complicated consideration in its proof. However it seems to be interesting in its own right. Gross' proof is very simple. We shall give several related results.

2. We need several known results.

LEMMA 1. [4]. *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{a_n\}_{n=1}^{\infty}$ such that all the roots of the equations $f(z) = a_n (n = 1, 2, \cdots)$ lie on a single straight line. Then $f(z)$ is a polynomial of degree at most two.*

This and the following lemma play an important role in the factorization theory.

LEMMA 2. [11]. *Let $F(z)$ be an entire function of finite order. Assume that $F(z) = f(g(z))$ holds with two transcendental entire functions f and g. Then the order ρ_f of f is equal to zero and $\rho_g \leqq \rho_F$.*

This result holds for meromorphic F. Indeed Edrei and Fuchs [5] proved the following.

LEMMA 3. [5]. *Let f be meromorphic of positive order, and let $g(z)$ be transcendental entire. Then $F(z) = f(g(z))$ is of infinite order.*