THE GENERALIZED PERRON INTEGRALS.*)

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Introduction. The notion of differentiation was generalized in many directions. Among them we consider the approximate derivative and Cesàro derivative. As the inverse of those derivatives, approximately continuous Perron integral and Cesàro-Perron integral are defined by J. C. Burkill [1], [2], using the Perron method.

In the definitions of these integrals, he assumed the contintity property of upper and lower functions. Recently S. Saks [3] defined the ordinary Perron integral without using any continuity property of the upper and lower functions, and proved the continuity of the indefinite integral. Hence it arises the problem, whether the notion of continuity of upper and lower functions are superfluous in the Burkill integrals or not.

We will, in this paper, answer this problem affirmatively. By this, the definition of the integrals becomes simply in some way. In §1 we define the approximately continuous Perron integral, or simply (AP)-integral and prove the approximate continuity of the indefinite (AP)-integral. In §2 we define the Cesàro-Perron integral, or simply (CP)-integral and prove the Cesàro continuity of the indefinite (CP)-integral.

1. The approximately continuous Perron integral or (AP)-integral.

Theorem 1.1. If measursble function $f(x)$ has a non-negative lower approximate derivate at each point of $[a, b]$, then $f(a) \leqq f(b)$.

Proof. Since the lower approximate derivate of $f(x)$ is non-negative at $x=a$, $A\underline{D}\, f(a) \geqq 0$, and then, for any small $\varepsilon (>0)$, the set

(1) $$S \equiv \{x \,|\, f(x) - f(x) \,|\, \geqq -\varepsilon (x-a)\}$$

has the point a as a point of density. For a given k $(0<k<1)$, we can find x_1 sufficiently near a, such that

$$f(x_1) - f(a) \geqq -\varepsilon (x_1 - a)$$

and that the set S has average density in (a, x_1) greater than k. Again starting from x_1 we can find x_2 such that

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