NOTES ON BANACH SPACE (VIII): A GENERALIZATION OF SILOV'S THEOREM.*)

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The set of all real-valued continuous functions defined on a compact Hausdorff space S forms a commutative Banach algebra R with respect to usual addition, product, scalar multiplication and the norm:

$$|x| = \sup_s |x(s)|.$$

Therefore, it is possible to introduce some notions of algebra with some modifications. For example, we mean by an *ideal* I a closed algebraic ideal in R and by a principal ideal $[x]$ the closure of the set of all elements xy where y runs through R. Moreover, we say, R is a principal ideal ring if and only if all its ideals are principal.

Under the above definitions, G. Silov [3] proved that R is a principal ideal ring if S is a compact metric space. In this note, we prove the converse theorem, which reads as follows:

Theorem. Banach algebra of real-valued continous functions on a compact Hausdorff space is a principal ideal ring if and only if the space is completely normal.

By a completely normal space S we mean a T_1-space satisfying one of the following three equivalent conditions:[1)]

1. S is normal and every its closed set is a G_δ-set.

2. For any closed set F of S, there exists a continuous function $x(s)$ in R such as

$$F = \{s \mid x(s) = 0\}.$$

3. For any two closed sets F and F' mutually disjoint, there is a non-negative continuous function $x(s)$ in R such that $|x| = 1$,

$$F = \left\{ s \mid x(s) = 0 \right\} \text{ and } F' = \left\{ s \mid x(s) = 1 \right\}.$$

The proof of the sufficiency of the theorem is almost similar to that of G. Silov in the case of a compact metric space S. But, for the sake of completeness, we give it in full. Firstly, we will prove the following lemma.

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1) The term is due to A. Komatu[2], where the equivalence of the conditions is also proved.