NOTES ON FOURIER ANALYSIS (XVII): THE INTEGRATED LIPSCHITZ CONDITION OF A FUNCTION AND FEJER MEAN OF FOURIER SERIES.*)

By

Sigeki Yano.

I. Let $f(x)$ be a function of period 2π satisfying the intgrated Lipschitz condition Lip (α, p) $(0<\alpha\leqq 1, p\geqq 1)$, that is

$$(1.1)\qquad \left(\int_0^{2\pi} |f(x+t)-f(x)|^p dx\right)^{1/p} = O(t^\alpha),$$

and let its Fouries seies be

$$(1.2)\qquad f(x) \sim a_0/2+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx).$$

If $f \varepsilon$ Lip (α, p) $(0<\alpha<1, p\geqq 1)$ and $\sigma_n(x, f)=\sigma_n(x)$ denotes the Fejér mean of (1. 2), then it will be easily seen that[1)]

$$(1.3)\qquad \left(\int_0^{2\pi} |f(x)-\sigma_n(x)|^p dx\right)^{1/p} = O(n^{-\alpha}).$$

This does not hold generally for $\alpha=1$, $p=1$, but we have

$$(1.4)\qquad \int_0^{2\pi} |f(x)-\sigma_n(x)| dx = O(n^{-1} \log n).$$

This will be seen by the following example. If we put

$$(1.5)\qquad f(x)=\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi-x}{2} \qquad (0\leqq x<2\pi),$$

then $f(x)$ belongs to Lip(1. 1)[2)]. Now

$$f(x)-\sigma_n(x) = \frac{1}{n}\sum_{k=1}^{n}\left\{ f(x)-s_k(x)\right\}$$

$$= \frac{1}{n}\sum_{k=1}^{n} \sin kx - \frac{1}{n}\sum_{k=1}^{n} \frac{\sin kx}{k} + \sum_{k=n+1}^{\infty} \frac{\sin kx}{k}$$

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1) G. H. Hardy and J.E. Littlewood, A convergence criteria for Fourier series, Math. Zeitschr. 28, (1928).

2) It is well known that if $f(x)$ is of bounded variation then $f(x)$ belongs to Lip(1.1). cf. loc. cit. Lemma 9.