NOTES ON FOURIER ANALYSIS (XXXIX): THEOREMS CONCERNING CESARO SUMMABILITY*)

By

SHIN-ICHI IZUMI AND GEN-ICHIRÔ SUNOUCHI

In this paper it is proved that, if

$$(1) \qquad \int_0^t \varphi_x(u)du = o\Big(t/\log\frac{1}{t}\Big), \quad \text{as} \quad t \to 0,$$

then the Fourier series of $f(t)$ is summable $(C, 1)$ at $t=x$, and if $0<\alpha<1$ and

$$(2) \qquad \int_0^t \varphi_x(u)du = o(t^{1/\alpha}), \quad \text{as} \quad t \to 0,$$

then the Fourier series of $f(t)$ is summable (C, α) at $t=x$. These theorems are known (Wang [7], [8]), but we give two kinds of proof. Each method is generalized to prove more general theorem. We prove that o in (1) and (2) cannot be replacd by O in these theorems.

§ **1.** THEOREM 1. *If*

$$(1) \qquad \int_0^t \varphi(u)du = o\Big(t/\log\frac{1}{t}\Big), \quad as \quad t \to 0,$$

where

$$\varphi(u) = \varphi_x(u) = \{f(x+u) + f(x-u) - 2f(x)\}/2,$$

then the Fourier series of $f(t)$ is summable $(C, 1)$ at $t=x$.

We prove this theorem in two ways, one using Young's function and the other using the Fejér kernel, respectively.

THE FIRST PROOF OF THEOREM 1. For $\alpha>0$, Young's function is defined by (Hobson [2] and Bosanquet [1])

$$\gamma_{1+\alpha}(u) = \int_0^1 (1-t)^\alpha \cos tu\, dt.$$

Then, as is well known, $\gamma_{1+\alpha}(u)$ and its derivative $\gamma'_{1+\alpha}(u)$ are bounded for $n\geqq 0$ and

$$(3) \quad \gamma_{1+\alpha}(u) \sim \frac{\Gamma(1+\alpha)}{u^{1+\alpha}} \cos\Big(u - \frac{\alpha+1}{2}\pi\Big) + O\Big(\frac{1}{u^{\alpha+2}}\Big) + O\Big(\frac{1}{u^2}\Big) \ (u\to\infty)$$

and $\gamma'_{1+\alpha}(u)$ has the behaviour of the derivative of the right hand side of (3) as $u\to\infty$. Especially, for $0<\alpha\leqq 1$,

$$(4) \qquad \gamma_{1+\alpha}(u) = O(1/u^{1+\alpha}) \quad (u\to\infty).$$

The necessary and sufficient condition that the Fourier series of $f(t)$ is summable $(C, 1)$ at $t=x$, is that

---

*) Received Apr. 3rd., 1950.