NOTES ON FOURIER ANALYSIS (XLVI): A CONVERGENCE CRITERION FOR FOURIER SERIES

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(Received October 30, 1950)

1. Introduction. The object of this paper is to generalize Young's convergence criterion for Fourier series. To simplify the writing, we shall suppose that the Fourier series

$$\varphi(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt$$

in question is that of an even periodic function which is integrable in the Lebesgue sense. Then Pollard [4] generalizes Young's test as follows.

THEOREM. *The Fourier series of $\varphi(t)$ converges at the point $t = 0$ to the value zero, provided that*

$$(1) \qquad \int_0^t \varphi(u)du = o(t), \qquad \text{as } t \to 0$$

and

$$(2) \qquad \int_0^t |d\{u\varphi(u)\}| = O(t), \qquad 0 \leqq t \leqq \eta.$$

On the other hand Hardy and Littlewood [1] proposed the problem, whether we can replace (1) and (2) by

$$(3) \qquad \int_0^t \varphi(u)du = o\left(t \Big/ \log \frac{1}{t}\right), \quad \text{as } t \to 0$$

and

$$(4) \qquad \int_0^t |d\{u^{\Delta}\varphi(u)\}| = O(t), \quad 0 \leqq t \leqq \eta,$$

for some $\Delta > 1$. Later Randels [5] proved that this is impossible. Concerning this problem we shall prove the following theorem.

THEOREM. *The Fourier series of $\varphi(t)$ converges at the point $t = 0$ to the value zero, provided that there is a $\Delta \geqq 1$ such that*

$$(5) \qquad \int_0^t \varphi(u)\, du = o(t^{\Delta}), \quad \text{as } t \to 0,$$

and

$$(6) \qquad \int_0^t |d\{u^{\Delta}\varphi(u)\}| = O(t), \qquad 0 \leqq t \leqq \eta.$$

2. Proof of Theorem. It is sufficient to prove that