GROUP REPRESENTATION AND BANACH LIMIT

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1. Introduction. Recently, B. de Sz. Nagy [3] proves the following theorem: In a Hilbert space H, any uniformly bounded (cyclic) group of linear operators T^n is similar to unitary one, or more precisely, if $|T^n| \leqq k$ for all n then there exists a self-adjoint operator B with $1/k \leqq B \leqq k$ such that $U = BTB^{-1}$ is unitary. If we consider in this theorem that T^n is a "representation" of the additive group of integers in H, then we can say that any bounded representation of the group in H is always similar to unitary one. Thus the theorem becomes a generalization of the well-known theorem concerning bounded matrix group-representation. Hence we will consider in this note the above cited de Sz. Nagy's theorem as the following problem:

(A) *Is every bounded (strong ly continuons) representation $T(g)$ of a topological group G in a Hilbert space H necessary similar to unitary one?*

In § 2, we shall discuss the above problem and partly answer it checking de Sz. Nagy's proof that the problem can be solved if the group G has a Banach limit (Proof can be carried out word for word following to de Sz. Nagy's). Thus the problem (A) is (partly) reduced to the following new problem:

(B) *Is it possible to define a Banach limit in the given group G?*

This problem will be considered in § 3, and it is reduced there to a fix-point theorem in the "state" space of all unifomly bounded continuous functions on G. Thus it is solved for some special cases which include both abelian and compact.

2. Theorem of B. de Sz. Nagy. Let us assume that G is a topological group and it has a *Banach limit* $\mathrm{Lim}_g\, x(g)$ for all uniformly bounded continuous functions $x(g)$ on G which satisfies the following conditions:

(1) $\mathrm{Lim}_g (\alpha x(g) + \beta y(g)) = \alpha\, \mathrm{Lim}_g\, x(g) + \beta\, \mathrm{Lim}_g y(g)$,

(2) $\mathrm{Lim}_g\, x(g) \geqq 0$ if $x(g) \geqq 0$,

(3) $\mathrm{Lim}_g\, x(g) = 1$ if $x(g) = 1$ for all g,

(4) $\mathrm{Lim}_g\, x(gh) = \mathrm{lim}_g\, x(g)$.

Furthermore, we may assume that there exists a uniformly bounded strongly continuous representation $T(g)$ of G in a Hilbert space H.

Under these circumstances we can solve the problem (A) following to B. de Sz. Nagy [3]. For the sake of completeness we shall repeat it under our notations.

Let $< x,y > = \mathrm{Lim}_g(T(g)x,\ T(g)y)$ for any pair x, y of H, then $< x,y >$