AN ELEMENTARY AND PURELY SYNTHETIC PROOF FOR THE DOUBLE SIX THEOREM OF SCHLÄFLI

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In one of my previous papers [43][1] I have given four elementary, purely projective and synthetic proofs of the double six theorem of Schläfli carried on within the scope of points, lines and planes only. In the present paper I will give a new, again purely synthetic proof of the same theorem, using only the Lemma 1 given below.

LEMMA 1. *Every straight line, which meets three straight lines out of given four all meeting three straight lines in skew positions, always meets the remaining fourth.*

The Lemma 2, which will also be used, follows immediately from the Lemma 1:

LEMMA 2. *When four straight lines a_1, a_2, a_3 and a_4 in skew positions are met by a straight line b_6, there exists necessarily another straight line b_5 different from b_6, which meets all of a_1, a_2, a_3 and a_4, provided that the row of points $b_6(a_1a_2a_3a_4)$ and the sheaf of planes $b_6(a_1a_2a_3a_4)$ are not projective.*

In the axiomatic theory of the projective geometry, the Lemma 1 is equivalent to the fundamental theorem of the projective geometry: "the rows of points $ABCD$ and $ABCD'$ are projective, when and only when D and D' are one and the same"; so that these are equivalent to the theorem of Pascal-Pappus as well as to the law of commutativity of the field (Körper), whereon the coordinates are defined. Hence it will be worth while to prove the double six theorem of Schläfli and the Lemma 2 using only the Lemma 1.

PROOF FOR LEMMA 1. Let the four straight lines meeting all of three straight lines b_1, b_5 and b_6 be a_2, a_3, a_4 and c respectively. I will show that a straight line d which meets all of a_2, a_3 and a_4 also meets c. If we denote, namely, the intersection points (b_5c) and (b_6c) by X and Y respectively, then among the sheaves of planes, the following relations hold:

$$d(a_2a_3a_4X) \overset{b_5}{\barwedge} b_1(a_2a_3a_4c) \overset{b_6}{\barwedge} d(a_2a_3a_4Y).$$

Hence the planes Xd and Yd are one and the same, so that d meets the joining line c of X and Y.

PROOF FOR LEMMA 2. Take any two straight lines f and g, which meet all of a_1, a_2 and a_3 and are different from b_6. Let the joining line of the

1) The number in the square brackets refers to the references at the end.