ON ORDER AND COMMUTATIVITY OF B^*-ALGEBRAS

M. FUKAMIYA, Y. MISONOU and Z. TAKEDA

(Received June 8, 1954)

The representation theory of (partially) ordered vector spaces has an application to the representation theory of commutative B^*-algebras. Kadison has treated this idea [2]. In this respect, we shall notice that the B^*-algebra with the decomposition property is necessarily commuative, which is a generalization of a commutativity theorem of Sherman [5] and might simplify the argument such as Kadison's when we apply the ordered vector space to the representation theory of B^*-algebras. Incidentally different proofs were obtained, which we shall state in the following. §1 is due to Misonou, §2 to Fukamiya and §3 to Takeda.

1. Theorem and its direct treatment. By a B^*-algebra, we mean a Banach algebra possessing a $*$-operation such as $\|x^*x\| = \|x\|^2$. It has recently been proved that every B^*-algebra can be represented as a uniformly closed, self-adjoint algebra of bounded operators on a suitable Hilbert space. Let A be a B^*-algebra and H, D be the set of all hermitian elements and positive hermitian elements in A respectively, then H is an archimedean ordered vector space by an order relation $a \leqq b$ in H as $b - a \in D$. We say a B^*-algebra A satisfies the *decomposition property*, originally due to F. Riesz, if for every a such as $0 \leqq a \leqq b + c$ with b and c positive, there exist positive a_1, a_2 such that $a = a_1 + a_2, a_1 \leqq b, a_2 \leqq c$. Then we shall prove

THEOREM 1. *A B^*-algebra A which has an identity e and satisfies the decomposition property is necessarily commutative.*

FIRST PROOF OF THEOREM. As a preparation, we notice that every projection p and hermitian operator a on a Hilbert space such that $0 \leqq a \leqq p$ satisfy $ap = pa$. For, by the assumption, we have $0 \leqq (1-p)a(1-p) \leqq 0$, which implies $a^{\frac{1}{2}}(1-p) = 0$, hence $a(1-p) = 0$ and $a = ap = pa$.

Since every element of A can be expressed as a linear combination of positive elements of A, it is sufficient to prove that $ab = ba$ for every pair of positive elements $a, b \leqq e$.

Let B be the B^*-subalgebra of A generated by a and e. Then B can be isomorphically represented to a ring $C(\Lambda_a)$ of all continuous function on the spetrum Λ_a of a. We denote by V the weak closure of an operator representation of B on a suitable Hilbert space.

Let $a(t)$ be the function corresponding to a by the function representation of B on Λ_a. Then $a(t)$ can be approximated at each point of Λ_a by a sequence $\{s_n(t)\}$ of step functions. This means there exists a sequence $\{s_n\}$ of linear combinations of projections in V which converges strongly to a. Hence, to prove the theorem it is sufficient to show that b is commutative with each