EXPECTATIONS IN AN OPERATOR ALGEBRA

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Introduction. Let A be a C^*-algebra having the identity. A mapping $\theta(x) = x^{\varepsilon}$ will be called an *expectation* of A if it satisfies

(0.1) $$(\alpha x + \beta y)^{\varepsilon} = \alpha x^{\varepsilon} + \beta y^{\varepsilon},$$

(0.2) $$x^{*\varepsilon} = x^{\varepsilon *},$$

(0.3) $$x \geq 0 \text{ implies } x^{\varepsilon} \geq 0,$$

(0.4) $$(x^{\varepsilon} y)^{\varepsilon} = x^{\varepsilon} y^{\varepsilon} = (x y^{\varepsilon})^{\varepsilon},$$

(0.5) $$1^{\varepsilon} = 1;$$

and will be called *abelian* if it satisfies moreover

(0.6) $$(xy)^{\varepsilon} = (yx)^{\varepsilon}.$$

Many known operations on C^*-algebras can be considered as expections:

EXAMPLE 1. If σ is a *state* (in the sense of I. E. Segal), i. e., a linear functional on A which is positive and normalized, then σ can be considered as an expectation of A which maps A into the field of scalar multiples of the identity: For, (0.1)–(0.3) and (0.5) are obvious and (0.4) follows from $\sigma(x\sigma(y)) = \sigma(x)\sigma(y)$. The trace of A is a *scalar* valued expectation which is abelian on A.

EXAMPLE 2. J. Dixmier's *centering* $\natural$ can be generalized in a C^*-algebra as an expectation of A into the center Z, which is abelian and

(0.7) $$x \in Z \text{ implies } x^{\natural} = x.$$

A (bounded) trace τ on a finite W^*-algebra can be considered the expectation of A which is the combination of a state and the centering, since

(0.8) $$\tau(x) = \tau(x^{\natural})$$

for any x. (Cf. also [5]).

For spaces of functions, the following examples exist:

EXAMPLE 3. Let A be the space of all continuous functions defined on $S \times T$ where S and T are compact spaces. Put

(0.9) $$x^{\varepsilon}(s, t) = \int x(s', t)ds'.$$

Then it is not hard to show that x^{ε} is an expectation of A, since

$$(x^{\varepsilon}y)^{\varepsilon} = \int y(s, t)\int x(s', t)ds'ds = \int x(s', t)ds' \int y(s, t)ds.$$

EXAMPLE 4. Let A be the space of bounded random variables on a